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GEOMETRIC APPROACH IN ROBOTIC SNAKE MOTION CONTROL
GEOMETRICKÉ POSTUPY V ŘÍZENÍ ROBOTICKÝCH HADŮ

MASTER'S THESIS
DIPLOMOVÁ PRÁCE

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Concise characteristic of the task:

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Recommended bibliography:

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Summary
This thesis deals with the description of controllability of a specific robotic snake named trident snake robot. This robot is classified as a nonholonomic system. The kinematics model is converted into the language of differential geometry and controlled by vector fields and their operation Lie bracket. Approximation of the controlling distribution is also considered. Next, vector field motions are described and also their combinations which provide basic planar surface motions (rotation and translation). Finally, these motions caused by vector fields are simulated in V-REP.

Abstrakt
Tato diplomová práce se zabývá popisem řiditelnosti specifického robotického hada, který se nazývá trident snake robot. Tento robot je řazen mezi neholonomní systémy. Model je převeden do jazyka diferenciální geometrie a řízen pomocí vektorových polí a operace na nich zavedené (Lieova závorka). Je také uvažována aproximace řídicí distribuce. Dále jsou formulovány pohyby hada ve směru vektorových polí a jejich kombinace, které zajišťují základní pohyby v prostoru (rotace a translace). Tyto pohyby jsou na závěr simulovány v prostředí V-REP.

Keywords
nonholonomic system, trident snake robot, Lie bracket, nilpotent approximation, V-REP, simulation

Klíčová slova
neholonomní systém, trident snake robot, Lieova závorka, nilpotentní aproximace, V-REP, simulace

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Abstrakt

Tato diplomová práce popisuje model robotického hada tzv. trident snake robota, pohybujícího se jen po rovinné ploše, z hlediska řiditelnosti. Tento model byl poprvé představen v článku [6]. Trident snake robot je zástupcem neholonomního systému, což zjednodušeně znamená, že tento mechanismus se nemůže pohybovat libovolným směrem ve svém stavovém prostoru. Tato skutečnost je vyjádřena neholonomními podmínkami, které určité pohyby robotu znemožňují. V případě trident snake robota jsou tímto omezením páry koleček na jeho nohou, které se nemohou natáčet a prokluzovat. Z těchto podmínek pak vychází odvození kinematických rovnic platících pro tento model.

Uvedeme tedy krátký popis mechanismu robota. Je sestrojen z těla ve tvaru rovnostranného trojúhelníka, na jehož vrcholech má připevněny nohy aktivními spoji, které jsou ovládány servomotorky. Poznamenejme, že zde mluvíme o zjednodušeném jednočlánkovém modelu, jelikož obecně je trident snake robot konstruován s libovolným počtem článků na každé noze. Na konci každé nohy (v obecném případě uprostřed článku) je pak umístěn pár pasivních koleček, které se nemohou otáčet a prokluzovat. Celý mechanismus je tedy řízen jen pomocí servomotorků, které natáčí nohy robota a způsobují pohyb celého robota. Důležitým předpokladem této omezující podmínky je skutečnost, že tření ve směru tečném ke každému článku je řádově větší než tření ve směru kolmém na článek, tedy pohyb celého mechanismu je realizován tzv. hadovitým pohybem.

Důležitou součástí této práce je zavedení pojmů diferenciální geometrie, popisující model trident snake robota. Zavádíme například varietu jako stavový prostor robota, vektorová pole a vektory na této varietě jako možné směry dalšího pohybu hada a Lieovu závorku jako operaci dvou vektorových polí, která definuje pole nové jako učitou kombinaci polí vstupujících do této operace. Dále je zaveden pojem řídicí distribuce a filtrace.

V dalším je také sestaven řídicí systém robota vycházející z kinematických rovnic. Řiditelnost systému je pak zkoumána z hlediska Chow-Rashevského věty, která říká, že pokud je stavový prostor spojitý a dimenze řídicí distribuce neholonomního systému je shodná s dimenzí stavového prostoru, je pak tento systém lokálně řiditelný. Tato věta tedy platí i pro trident snake robota, jehož řídicí distribuce odpovídá Lieově algebře složené ze tří základních vektorových polí g_1, g_2, g_3 a jejich Lieových závorek. Má tedy tvar $\Delta(q)$ =span$\{g_1, g_2, g_3, [g_1, g_2], [g_1, g_3], [g_2, g_3]\}$ a její dimenze je rovna 6. Jelikož je trident snake robot popsán 6 základními souřadnicemi $x, y, \theta, \phi_1, \phi_2, \phi_3$, je dimenze variety taktéž rovna 6.

Přibližme nyní význam jednotlivých souřadnic stavového prostoru. Souřad-

nice x, y popisují umístění robota na ploše a θ pak natočení celého těla robota. Zbývající tři souřadnice pak popisují úhel natočení jednotlivých nohou vzhledem k výchozí poloze.

Další část práce pak hledá vhodnou linearizaci systému, který by jej zjednodušil. Jelikož je náš systém neholonomní, nelze použít klasickou linearizaci dynamického systému pomocí rovnovážných bodů a Jacobiho matice. Náš nelineární řídicí systém lze linearizovat ve smyslu sub-Riemanovy vzdálenosti. Vzdálenost dvou bodů pak můžeme chápat jako minimální čas, který neholonomní systém potřebuje k přesunu mezi těmito dvěma body jen ve směrech řídicích vektorových polí tohoto systému.

Pro zavedení vhodné aproximace dále uvádíme pojmy jako je stupeň neholonomity funkce a vektorového pole v bodě a demonstrujeme jejich výpočty na jednoduchých příkladech. Dalším krokem je pak zavedení nového systému souřadnic, který odpovídá směrům jednotlivých řídicích vektorových polí. V těchto souřadnicích pak vyjádříme všechna řídicí vektorová pole trident snake robota a rozvineme je do Taylorových řad. Z těchto řad pak bereme jejich první člen, který prohlásíme za aproximaci systému. Důležitou poznámkou je, že Lieova algebra generovaná třemi základními aproximovanými poli $\hat{g}_1, \hat{g}_2, \hat{g}_3$ je nilpotentní, jelikož operace Lieova závorka vyššího stupně než 1 je nulová. Tato skutečnost je v této práci ověřována v softwaru Maple.

Poslední část práce se zabývá hledáním způsobu, jakým lze trident snake robota řídit. Nejdříve je zde uveden přehled pohybů ve směru jednotlivých řídicích vektorových polí a jejich Lieových závorek. Některými kombinacemi těchto polí pak získáme základní pohyby na rovině. A to zejména rotace kolem osy z a pohyb ve směru os souřadného systému x-y.

Pro simulaci těchto pohybů je používáno prostředí V-REP, které zohledňuje fyzikální vlastnosti i vlivy prostředí. Je nutné poznamenat, že k simulaci pohybu ve směru Lieovy závorky dvou vektorových polí je použit periodický input, který do výsledků přináší jistou nepřesnost. Pohyby ve směrech Lieových závorech všech základních vektorových polí jsou tedy prezentovány včetně časového vývoje na příslušných grafech a obrázcích.

Závěrem práce ještě diskutujeme možné uplatnění aproximovaného modelu pro simulaci v prostředí V-REP. Důsledkem mnoha nepřesností při simulaci pohybů docházíme k závěru, že v tomto prostředí nelze hodnotit odlišnosti původního a aproximovaného modelu. Pro přesné strovnání těchto modelů odkazujeme na příslušnou literaturu.

I declare that I worked on this thesis on my own under the guidance of my supervisor and that I used only sources mentioned in the list of literature.

Bc. Jana Vechetová

I would like to thank my supervisor doc. Mgr. Petr Vašík, PhD. for his valuable advices, helpful tips and encouraging words.

Bc. Jana Vechetová

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Introduction

This thesis Geometrical approach in robotic snake motion control introduces a simplified model trident snake robot. This mechanism is further studied in details.

In the first chapter we introduce a model of a trident snake robot with an appropriate kimematics which we derive from nonholonomic constraints. These constraints are derived by non–slip and non slide sideways condition for the wheels of the robot.

In Chapter 2, we introduce some elementary notions from differential geometry (manifold, tangent space, vector field, . . .) and interpret them for our trident snake model. We provide an illustrating example of a computation of a Lie bracket which plays significant role in the following control theory. This chapter concludes a formulation of Frobenius theorem.

Chapter 3 returns back to the trident snake control. We provide a control model based on a controlling vector fields and also a condition for ensuring a local controllability is introduced. Thus we say that a trident snake is locally controllable thanks to Chow–Rashevsky theorem.

In Chapter 4 we try to find a suitable linearization for our nonholonomic system. Therefore we recall several notions related to this topic and provide examples for better understanding. In this part we also introduce new coordinates called privileged ones which help in a procedure of searching a suitable approximation. At the end of a chapter a first–order approximation is found as a nilpotent associated with the new coordinates.

Chapter 5 presents a basic motions described by controlling vector fields and their Lie brackets which generate a controllability Lie algebra. Not only vector field motions are discussed. We also compose them to achieve translation and a rotation of the robot.

Chapter 6 introduces results of a simulation in a software called V–REP which includes physical properties of the model and environment. Despite inaccuracies caused by several influences we provide an overview of Lie bracket vector fields motions realized by periodic input. At the end of the Chapter we compare the original trident snake model with its nilpotent approximation.

1 Trident snake

This chapter is based on [6] and [5].

1.1 Model of the robot

In this work we deal with a three-headed snake robot moving on the planar surface which is called a trident snake robot. The mechanism was introduced in [6]. It is composed of a body in the shape of an equilateral triangle with circumscribed circle of radius r and three branches of serial links (also called legs) which are connected to the root block via actuated joints at the vertices of the triangle. There is also an actuator between each pair of links. Each link has a pair of passive wheels at the center, which is assumed not to slip, nor slide sideways. It provides an important snake-like property that the ground friction in the direction perpendicular to the link is considerably higher than the friction of a simple forward move.

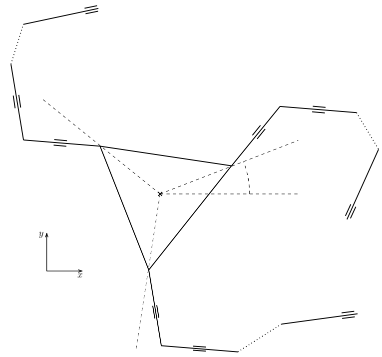

Figure 1:

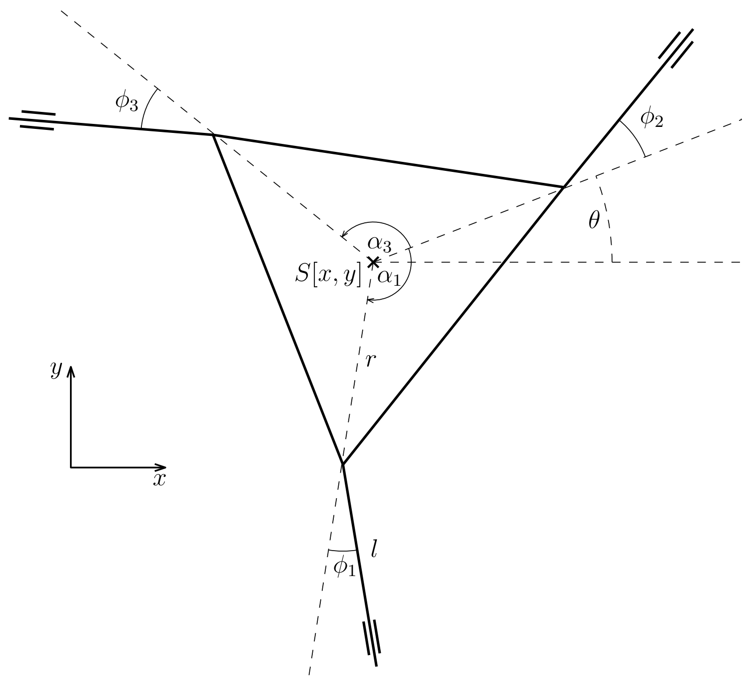

Figure 2: Trident snake robot, 1–link model

In our case we assume that each leg has only one link of the length $2l$, therefore the distance between a wheel and an adjacent joint is same as the radius $r = l = 1$. Thus we can consider this model with legs of length $l = 1$ and a pair of wheels at their ends.

To describe the actual position of a trident snake robot we need the set of 6 generalized coordinates

$$q = (x, y, \theta, \phi_1, \phi_2, \phi_3) =: (x_1, x_2, x_3, x_4, x_5, x_6),$$

where coordinates (x, y) represent the position of the center S of the robot with respect to a fixed coordinate system. The orientation of the mechanism is represented by the angle θ and the last three coordinates represent the rotation of the appropriate leg ϕ_i. Thus we have the configuration vector $\mathbf{w} := (x, y, \theta)^T$ of the robot and the shape vector $\phi := (\phi_1, \phi_2, \phi_3)^T$. Hence the configuration space is (a subspace of) $\mathbb{R}^2 \times \mathbb{S}^1 \times (\mathbb{S}^1)^3$. Note that a fixed coordinate system x, y is attached.

1.2 Kinematics

Now we derive the kinematics description of the trident snake robot. Let $x_i, y_i,\ i = 1, 2, 3$, denote the wheel positions. We have θ as the absolute orientation of the model and x, y the coordinates of the triangle centre. Then

$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} x + \cos(\alpha_i + \theta) + \cos(\alpha_i + \theta + \phi_i) \\ y + \sin(\alpha_i + \theta) + \sin(\alpha_i + \theta + \phi_i) \end{pmatrix}, \tag{1}$$

where α_i is the i–th central angle within the triangle. These angles are given by the construction of the robot.

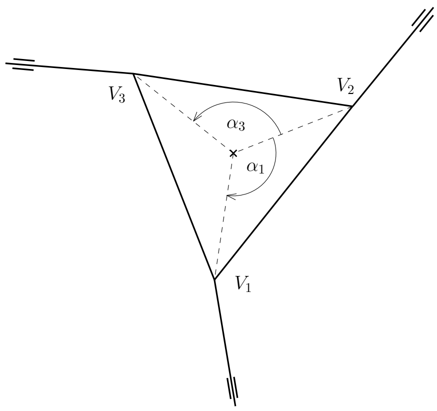

Figure 3: Construction of the mechanism

$$\alpha_1 = -\frac{2}{3}\pi \qquad \alpha_2 = 0 \qquad \alpha_3 = \frac{2}{3}\pi$$

The following three constraints are given by the non–slip and non–slide assumption on the wheels:

$$\dot{x}_i \sin(\alpha_i + \theta + \phi_i) = \dot{y}_i \cos(\alpha_i + \theta + \phi_i), \quad i = 1, 2, 3. \tag{2}$$

Differentiating the equations (1) and substituting into (2) we obtain the dynamical system

$$\begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{pmatrix} = \begin{pmatrix} \sin(\theta + \alpha_1 + \phi_1) & -\cos(\theta + \alpha_1 + \phi_1) & -(1 + \cos\phi_1) \\ \sin(\theta + \alpha_2 + \phi_2) & -\cos(\theta + \alpha_2 + \phi_2) & -(1 + \cos\phi_2) \\ \sin(\theta + \alpha_3 + \phi_3) & -\cos(\theta + \alpha_3 + \phi_3) & -(1 + \cos\phi_3) \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix}.$$

This system can be transformed (extracting the rotation matrix) to the form in which parameter θ is eliminated and thus it corresponds to non–inertial frame of reference

$$\dot{\boldsymbol{\phi}} = A(\phi) R_\theta^T \dot{\mathbf{w}}$$

$$\begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{pmatrix} = \begin{pmatrix} \sin(\alpha_1 + \phi_1) & -\cos(\alpha_1 + \phi_1) & -(1 + \cos\phi_1) \\ \sin(\alpha_2 + \phi_2) & -\cos(\alpha_2 + \phi_2) & -(1 + \cos\phi_2) \\ \sin(\alpha_3 + \phi_3) & -\cos(\alpha_3 + \phi_3) & -(1 + \cos\phi_3) \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix}.$$

2 Differential geometry

In this section we recall some elementary notions of differential geometry which are used in this text. Some of these notions will be specified for our particular model of a trident snake robot. For further details, proofs and consequences see [3], [4], [8] and [12].

2.1 Smooth functions and mappings

Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}$ be a function. We say that f is *r-times differentiable* or *function of the class C^r* if the function f has continuous partial derivatives of order r at all points of U. Function of the class C^∞ is called a smooth function.

Let $U \subset \mathbb{R}^n$ be an open set with $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$. Consider $V \subset \mathbb{R}^k$ another open set and $y = (y^1, \ldots, y^k) \in \mathbb{R}^k$. The mapping $f : U \to V$ is given by a k-tuple of functions $f^1, \ldots, f^k$ as

$$\begin{aligned} y^1 &= f^1(x^1, \ldots, x^n), \\ &\ldots \\ y^k &= f^k(x^1, \ldots, x^n). \end{aligned}$$

Functions $f^1, \ldots, f^k$ are called the *components of mapping f*. We also write

$$y^p = f^p(x^i), \qquad i = 1, \ldots, n,\ p = 1, \ldots, k.$$

We say that f is differentiable mapping of the class C^r if all the components are functions of the class C^r, $r = 1, \ldots, \infty$. Mapping of the class C^∞ is called a smooth mapping.

Now we recall very important notion of the *Jacobi matrix* which is related to mappings. Indeed, the matrix $\left(\frac{\partial f^p(a)}{\partial x^i}\right)$ is called the *Jacobi matrix* of mapping f in $a \in U \subset \mathbb{R}^n$. In the case $k = n$, the determinant of the mapping f in a is called the *Jacobian* and is denoted by $J = \det\left(\frac{\partial f^p(a)}{\partial x^i}\right)$.

Definition 2.1. Let $U, V \subset \mathbb{R}^n$ be open sets. Bijective mapping $f : U \to V$ is called a *diffeomorphism of the class C^r* if f and also the inverse mapping $f^{-1} : V \to U$ are of the class C^r, $r \geq 1$.

We can interpret a diffeomorphism $f : U \to V$ as a curvilinear coordinate system on U. For every $a \in U$ there exists an n–tuple of numbers $f(a) = (f^1(a), \ldots, f^n(a))$ which can be understood as a coordinate form of a. This fact is demonstrated in Figure 4.

Proposition 2.2. *If $f : U \to V$ is a diffeomorphism then the Jacobian* $\det\left(\frac{\partial f^i(a)}{\partial x^j}\right) \neq 0$ *for all $a \in U$.*

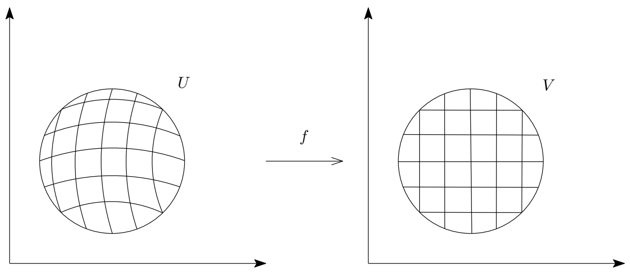

Figure 4: Curvilinear coordinates

2.2 Curves

In this section we want to recall some basic notions related to curves which will be later generalized to manifolds.

In general, there exist two elementary approaches to the notion of a curve. In geometry, we usually understand a curve to be a particular set of points within a plane (generally within n-dimensional Euclidean space $\mathbb{E}_n$). On the other hand, in mathematical analysis a curve often represents a graph of a smooth function which describes the trajectory of a point motion.

The dynamical approach can be specified as follows: consider an open interval $I \subset \mathbb{R}$ whose elements represent time values. A mapping $f : I \to \mathbb{E}_n$ is called a motion in $\mathbb{E}_n$. Sometimes we use a path instead of motion. If a basis is chosen in $\mathbb{E}_n$ then $f(t) = (f^1(t), f^2(t), \ldots, f^n(t))$ is a n-tuple of real functions therefore the motion can be indentified with a vector valued function $f : I \to \mathbb{R}^n$. The derivative of this vector valued function is then the motion velocity and we write $f' = \frac{\mathrm{d}f}{\mathrm{d}t}$. We say that the motion $f(t)$ is of the class C^r if all its components $f^i(t)$ have derivatives up to order r. We can easily see that the previous definition does not depend on the choice of the basis.

Next, we say that a motion f is *regular* if $f' \neq \overrightarrow{0}$ for all $t \in I$. It means that the regular motion has a velocity different from zero at any point. Motion f is called *simple* if the condition $t_1 \neq t_2 \implies f(t_1) \neq f(t_2)$ holds. Which means that the trajectory does not contain any self intersections.

Definition 2.3. A set $\mathcal{C} \subset \mathbb{E}_n$ is called a *simple curve* of class C^r if there exists such a simple regular motion $f : I \to \mathbb{E}_n$ of class C^r that $\mathcal{C} = f(I)$. A mapping f is then called a *parameterization* of a simple curve $f(I)$.

Proposition 2.4. *Mappings $f(t) : I \to \mathbb{E}_n$ and $g(\tau) : J \to \mathbb{E}_n$ are parameterizations of the same simple curve $\mathcal{C}$ of the class C^r if and only if there exists a bijection $\varphi : J \to I$, $t = \varphi(\tau)$ of the class C^r such that for all $\tau \in J$*

the assertion $\frac{d\varphi}{d\tau} \neq 0$ holds and $g(\tau) = f(\varphi(\tau))$. Function φ is then called a reparameterization or a parameter transformation of a curve $\mathcal{C}$.

Definition 2.5. The set $\mathcal{C} \subset \mathbb{E}_n$ is called a *curve of class C^r* if for any point $p \in \mathcal{C}$ there exists a neighbourhood U_p in $\mathbb{E}_n$ such that $\mathcal{C} \cap U_p$ is a simple curve of class C^r. The parameterization of the intersections $\mathcal{C} \cap U_p$ is called a *local parametrization of a curve $\mathcal{C}$.*

If $f : I \to \mathbb{E}_n$ is some local parameterization of a curve $\mathcal{C}$ then the line determined by the point $f(t_0) \in \mathcal{C}$ and a vector $f'(t_0)$ is called a tangent line of curve $\mathcal{C}$ at point $f(t_0)$.

Definition 2.6. We say that two curves $\mathcal{C}, \overline{\mathcal{C}} \subset \mathbb{E}_n$ at a common point $p \in \mathcal{C} \cap \overline{\mathcal{C}}$ have *the contact of order k* if there exist their local parameterizations $f(t)$ and $\overline{f}(t)$, $f(t_0) = \overline{f}(t_0) = p$, such that $\frac{d^i f(t_0)}{dt^i} = \frac{d^i \overline{f}(t_0)}{dt^i}$ for all $i = 1, \ldots, k$.

Clearly, two curves have the contact of order 1 at a common point if and only if they share the common tangent line at that point. It is also true that the tangent line is the only line having the contact of order 1 with a curve at a particular point.

Quite analogously, we define a surface. We start with a notion of a simple surface and its parameterization. Then the general surface is defined as a subset of $\mathbb{E}_3$ which can be locally parameterized around an arbitrary point of this surface.

2.3 Manifolds

Now let us introduce a generalization of the notions of curves and surfaces which is called a manifold. The main idea can be shown as an analogy of the Earth cartography. As we know, a sphere can not be described completely by one planar chart but it can be mapped as a set of charts ordered into an atlas. Strictly speaking, any point of a sphere has a neighbourhood such that it can be uniquely mapped on a subset of a plane. We use the notions of charts and atlas even in the following text, where we define a manifold precisely.

First, let us recall that a topological space is called Hausdorff if its any two points can be separated by two open disjoint sets. A homeomorphism is a continuous bijective mapping whose inverse is also continuous.

Definition 2.7. *An n–dimensional topological manifold* is a Hausdorff space M with a countable basis which is locally homeomorphic to $\mathbb{R}^n$, i.e. for every point $x \in M$ there exists its open neighbourhood $U \subset M$ and a homeomorphism $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$.

The pair (U, φ) is called a *local chart* and a system of charts $(U_\alpha, \varphi_\alpha), \alpha \in I$ on M such that U_α cover whole M is called an *atlas.* The demand on the countable basis guarantees that a finite or countable system of charts covering whole M can be chosen.

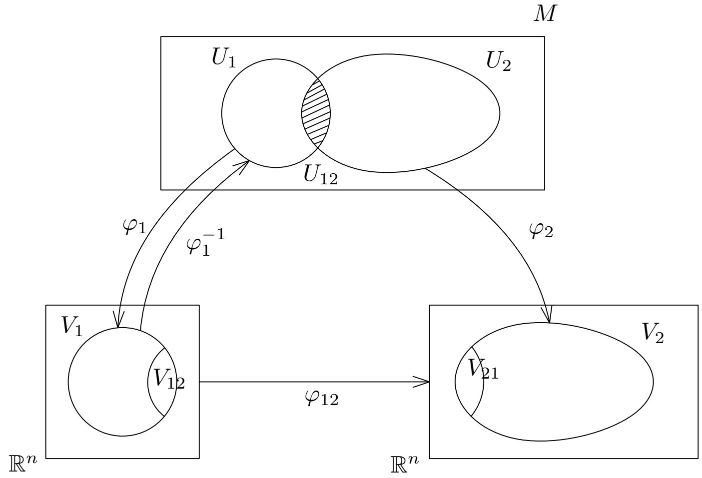

Figure 5: Chart changing mapping

Two local charts (U_1, φ_1), (U_2, φ_2) induce a mapping $\varphi_{12} := \varphi_1 \circ \varphi_2^{-1} : \varphi_2(U_1 \cap U_2) \to \varphi_1(U_1 \cap U_2)$ between two subsets of $\mathbb{R}^n$ called *the chart changing mapping.*

We say that an atlas $(U_\alpha, \varphi_\alpha)$ of a manifold M is of class C^k, if all chart changing mappings $\varphi_{\alpha\beta}$ are diffeomorphisms of class C^k (i.e. $\varphi_{\alpha\beta}$ is a bijection of class C^k such that its inverse is also of class C^k).

A chart (U_0, φ_0) is called compatible with the atlas $(U_\alpha, \varphi_\alpha)$ of class C^k if any chart changing mapping $\varphi_{0\alpha}$ is a diffeomorphism of class C^k. Atlas $(U_\alpha, \varphi_\alpha)$ of class C^k is called complete if it contains all compatible charts.

Definition 2.8. A differentiable manifold of class C^k is a topological manifold M with a complete atlas of class C^k.

A mapping φ from a local chart (U, φ) is given by an n–tuple of functions $(\varphi^1, \ldots, \varphi^n)$ denoted by $(x^1, \ldots, x^n)$ or (x^i) that are called local coordinates of a manifold M, the set U is called a coordinate neighbourhood.

We say that a mapping $f : M \to N$ between two manifolds is of class C^k if for any $x \in M$ and any chart (W, ψ) on N such that $f(x) \in W$, there exists a chart (U, φ) on M such that $x \in U$ and the mapping $\psi \circ f \circ \varphi^{-1}$ is of class C^k. A mapping $\psi \circ f \circ \varphi^{-1}$ is called a coordinate form of mapping f. If (y^p) are local coordinates on N, this coordinate form is $y^p = f^p(x^1, \ldots, x^n)$. Analogically, we define a function $f : M \to \mathbb{R}$ of class C^k. In the following

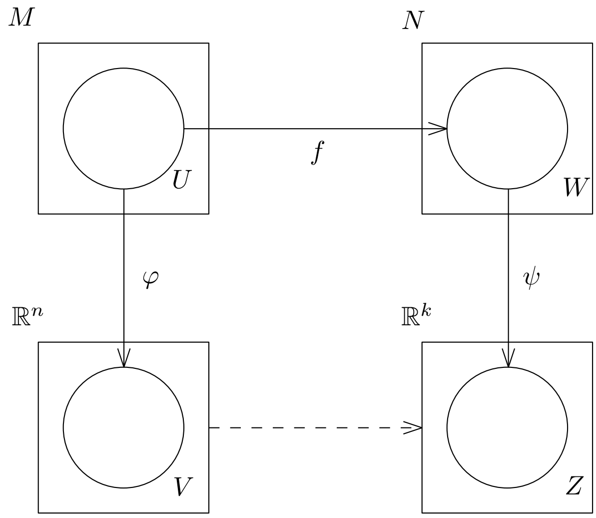

Figure 6: Mapping between manifolds

text, we assume all manifolds (mappings, functions) to be of class C^∞ and will be called smooth.

For our model of a trident snake robot, a manifold M represents a state space determined by coordinates $q = (x, y, \theta, \phi_1, \phi_2, \phi_3)$. The manifold is 6–dimensional thus each point of the manifold M can be represented by local coordinates (x^i) in a space $\mathbb{R}^6$.

2.4 Tangent bundle

We recall that the tangent vectors of a surface $\mathcal{S} \subset \mathbb{E}_3$ at a point $x \in \mathcal{S}$ are defined as the tangent vectors of such curves on $\mathcal{S}$ that contain the point x. We use this idea to establish the notion of a tangent vector to a manifold M. A smooth mapping $f : I \to M$ is called a path on a manifold M. In the following text we assume that the interval I contains zero.

Definition 2.9. We say that two paths $f, g : I \to M$ satisfying $f(0) = g(0) = a$ have a contact at a point $a \in M$ if there exists a coordinate neighbourhood U of a point a with local coordinates (x^i) such that

$$\frac{d(x^i \circ f)(0)}{dt} = \frac{d(x^i \circ g)(0)}{dt}.$$

Clearly, the above definition is independent on the choice of local coordinates. The equivalence class of paths $f(t)$ on M satisfying $f(0) = a$ and having a contact at $a \in M$ is called a tangent vector of a manifold M at a point a and we denote it as

$$t_a = \frac{df(0)}{dt}.$$

The elements

$$\xi^i := \frac{d(x^i \circ f)(0)}{dt}$$

are then called the coordinates of a vector t_a in local coordinates (x^i).

Now let explain the meaning of this notion for our model. Tangent vector t_a of a manifold M at an arbitrary point $a \in M$ can be understood as a direction of possible motion of the mechanism. In other words, if the model is in the state $a \in M$, the tangent vector t_a determines in which ways the model can be changed from this particular state.

Now we define a derivative of a function in the direction of tangent vector. If $\varphi = \varphi(x^i) : M \to \mathbb{R}$ is an arbitrary smooth function on U then

$$t_a\varphi := \frac{d(\varphi \circ f)(0)}{dt} = \sum_{i=1}^{n} \frac{\partial\varphi(a)}{\partial x^i}\frac{d(x^i \circ f)(0)}{dt} = \sum_{i=1}^{n} \frac{\partial\varphi(a)}{\partial x^i}\xi^i. \tag{3}$$

The value $t_a\varphi$ defined by (3) is called a derivative of a function φ in the direction of vector t_a. We can see that for arbitrary smooth functions $\varphi, \psi : M \to \mathbb{R}$ defined on the neighbourhood of a point a the following holds:

$$t_a(r\varphi+s\psi) = rt_a\varphi+st_a\psi, \quad t_a(\varphi\cdot\psi) = \varphi(a)\cdot t_a\psi+\psi(a)\cdot t_a\varphi, \quad r, s \in \mathbb{R}. \tag{4}$$

Consequently, it is possible to define a tangent vector as an operator t_a which assigns a real number $t_a\varphi$ to a function $\varphi : M \to \mathbb{R}$ and which satisfies (4). An example of a tangent vector is an operator $\left(\frac{\partial}{\partial x^i}\right)_a$ which to a function φ assigns its derivative with respect to x^i at a point a.

Definition 2.10. The set T_aM of all tangent vectors of a manifold M at a point a is called the tangent space of M at a point a.

Then T_aM is n–dimensional vector space whose basis may be formed for example by tangent vectors $\left(\frac{\partial}{\partial x^i}\right)_a$.

Then we see that if our model of trident snake is in a state $a \in M$, tangent space of M at a point a is a set of all directions in which the model can be changed from this state a.

The disjoint union of tangent spaces at every $a \in M$ is denoted by

$$TM := \bigcup_{a \in M} T_aM.$$

Then TM is $2n$–dimensional differentiable manifold called the tangent space. Hence we get a tangent bundle denoted by $TM \to M$ with a natural projection $p : TM \to M$ which to a tangent vector $t_a \in T_aM$ assigns the contact point a.

For our aim is important to define the inverse direction of the mapping. Thus we want to define a notion which to a point of a manifold assigns a tangent vector. In the next section we will see that a vector field meets this.

2.5 Vector fields

Definition 2.11. Let $TM \to M$ be the tangent bundle of a manifold M. By a vector field on M we understand a smooth mapping $X : M \to TM$ which to any point $a \in M$ assigns a tangent vector $X(a) \in T_aM$.

If (x^i) are local coordinates on the neighbourhood U of a point $x \in M$ then the vector field X can be expressed in the form

$$X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i},$$

where $X^i = X^i(x)$ are smooth functions defined on U.

We can imagine a vector field on a state space of a trident snake robot as a mapping providing that the direction of a motion is defined in each point of the manifold.

For any smooth function $f : M \to \mathbb{R}$, it is also possible to define its derivative $Xf : M \to \mathbb{R}$ along the vector field X by $(Xf)(a) = X(a)f$ where the right hand side stands for a derivative in the direction of a vector $X(a) \in T_aM$ as in (4).

In local coordinates we have

$$Xf = \sum_{i=1}^{n} X^i(x) \frac{\partial f}{\partial x^i} \; .$$

It turns out that the set $\chi(M)$ of all vector fields on M can be identified with the space of all derivatives of the algebra of smooth functions $C^\infty(M, \mathbb{R})$, i.e. with $\mathbb{R}$–linear operators $D : C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ satisfying $D(fg) = D(f)g + fD(g)$.

Just to remind we understand algebra as a vector space V (it has a binary operation addition and modular operation multiplication by an element from a field) with a bilinear operation $m : V \times V \to V$.

Definition 2.12. A path $f : I \to M$ is called an integral curve of a vector field X if the vector $X(f(t))$ is tangent to f at $f(t)$ for any $t \in I$, i.e.

$$\frac{df(t)}{dt} = X(f(t)) \quad \forall t \in I.$$

Any integral curve is thus a solution of an antonomous system of ODEs

$$\frac{dx^i}{dt} = X^i(x^1, \dots, x^n).$$

We remind that system of ODEs is called autonomous if it does not explicitly contain the independent variable t. We also know that the solution of autonomous system of ODEs is independent of the time at which the initial conditions are applied.

From the theory of ODEs it follows that for any point $x \in M$ there exists an open interval I_x containing 0 and an integral curve $f_x : I_x \to M$ of a vector field X such that $f_x(0) = x$. If the interval I_x is maximal then f_x is unique.

Furthermore, from the existence theorem for differential equations follows that the set $\mathcal{D}X := \bigcup_{x \in M} I_x \times \{x\} \subset \mathbb{R} \times M$ is open and a mapping

$$\mathcal{F}l^X : \mathcal{D}X \to M \quad \text{defined by} \quad \mathcal{F}l^X(t, x) = f_x(t)$$

is smooth. The mapping $\mathcal{F}l^X$ is called the flow of a vector field X.

2.6 Lie bracket

For any pair of vector fields X, Y on M there exists a unique vector field $[X, Y]$ on M such that for any function f on M the assertion

$$[X, Y]f = X(Yf) - Y(Xf)$$

holds. The vector field $[X, Y]$ is called a Lie bracket of vector fields X, Y. In the coordinate form we have

$$[X, Y] = \sum_{i,j=1}^{n} \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}.$$

For every $k, l \in \mathbb{R}$ and any vector fields X, Y, Z on M the following equalities hold

$$\begin{aligned} [kX + lY, Z] =& k[X, Z] + l[Y, Z], \qquad \text{(linearity)} \\ [X, Y] =& -[Y, X], \qquad \text{(anti–symmetry)} \\ [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] =& 0 \qquad \text{(Jacobi identity).} \end{aligned} \tag{5}$$

Example Now we show the computation of Lie bracket on simple example. Let X, Y be vector fields on two dimensional manifold with coordinates x_1, x_2 of the following form

$$\begin{aligned} X &= X^1 \frac{\partial}{\partial x^1} + X^2 \frac{\partial}{\partial x^2}, \\ Y &= Y^1 \frac{\partial}{\partial x^1} + Y^2 \frac{\partial}{\partial x^2}. \end{aligned}$$

Their Lie bracket is computed in the following way

$$\begin{aligned} [X, Y] = XY - YX = & \left(X^1 \tfrac{\partial}{\partial x^1}, X^2 \tfrac{\partial}{\partial x^2}\right) \left(Y^1 \tfrac{\partial}{\partial x^1}, Y^2 \tfrac{\partial}{\partial x^2}\right) - \\ & - \left(Y^1 \tfrac{\partial}{\partial x^1}, Y^2 \tfrac{\partial}{\partial x^2}\right) \left(X^1 \tfrac{\partial}{\partial x^1}, X^2 \tfrac{\partial}{\partial x^2}\right) = \\ & = X^1 \frac{\partial Y^1}{\partial x^1} \frac{\partial}{\partial x^1} + X^1 \frac{\partial Y^2}{\partial x^1} \frac{\partial}{\partial x^2} + X^2 \frac{\partial Y^1}{\partial x^2} \frac{\partial}{\partial x^1} + X^2 \frac{\partial Y^2}{\partial x^2} \frac{\partial}{\partial x^2} - \\ & - Y^1 \frac{\partial X^1}{\partial x^1} \frac{\partial}{\partial x^1} - Y^1 \frac{\partial X^2}{\partial x^1} \frac{\partial}{\partial x^2} - Y^2 \frac{\partial X^1}{\partial x^2} \frac{\partial}{\partial x^1} - Y^2 \frac{\partial X^2}{\partial x^2} \frac{\partial}{\partial x^2} = \\ & = \left(X^1 \tfrac{\partial Y^1}{\partial x^1} + X^2 \tfrac{\partial Y^1}{\partial x^2} - Y^1 \tfrac{\partial X^1}{\partial x^1} - Y^2 \tfrac{\partial X^1}{\partial x^2}\right) \frac{\partial}{\partial x^1} + \\ & + \left(X^1 \tfrac{\partial Y^2}{\partial x^1} + X^2 \tfrac{\partial Y^2}{\partial x^2} - Y^1 \tfrac{\partial X^2}{\partial x^1} - Y^2 \tfrac{\partial X^2}{\partial x^2}\right) \frac{\partial}{\partial x^2}. \end{aligned}$$

Example Let us show a graphic meaning of the Lie bracket. We can understand this operation as a specific combination of two vector fields (later we will mention a periodic input to simulate it).

For two vector fields g_1, g_2 and the initial point q_0 the Lie bracket is com-

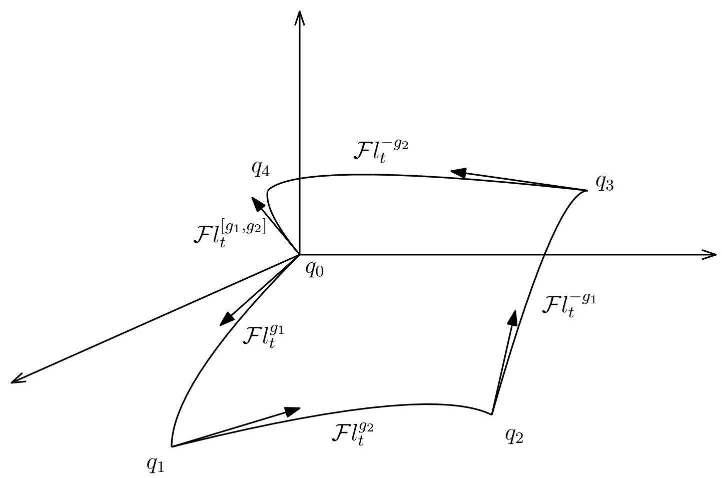

Figure 7: Lie bracket $g_{12} = [g_1, g_2]$

posed by a flow of a vector field g_1 per time t (we are in a point q_1), and flows $\mathcal{F}l_t^{g_2}, \mathcal{F}l_t^{-g_1}, \mathcal{F}l_t^{-g_2}$ therefore we finish at point q_4. Thus we see that the combination gives us a new vector field g_{12}.

Note that the set of all vector fields $\chi(M)$ on a manifold M together with the Lie bracket form a Lie algebra which, generally, is defined as a vector space V over a field F together with a binary operation denoted by $[\cdot,\cdot] : V \times V \to V$ defined by $[X,Y] = XY - YX$ for $X, Y \in V$, for which the equalities (5) hold.

2.7 Frobenius theorem

A rule which to every $x \in M$ assigns a k-dimensional linear subspace $S(x) \subset T_xM$ is called a *k-dimensional distribution* S on the manifold M.

We say that the vector field X on M belongs to the distribution S if $X(x) \in S(x)$ for all $x \in M$.

Definition 2.13. Distribution S is called *smooth* if for every $x \in M$ there exist such a neighbourhood U and k smooth vector fields $X_1, \dots, X_k$ on U such that the vectors $X_1(x), \dots, X_k(x)$ form the basis of $S(x)$ for all $x \in V$.

In the following text we consider smooth distributions only.

Definition 2.14. k-dimensional submanifold $N \subset M$ is called *integral manifold of a distribution* S if $T_xN = S(x)$ for every $x \in N$. Then also the distribution S is called *integrable* if for every $x \in M$ there exists an integral manifold of the distribution S passing through the point x.

Now we define important property of a distribution which says that the operation Lie bracket is closed on the distribution.

Definition 2.15. The distribution S is called *involutive* if it has the following property: if two vector fields X_1, X_2 defined on the same open set $U \subset M$ belong to S then also their bracket $[X_1, X_2]$ belongs to S.

Let us formulate significant Frobenius theorem.

Theorem 2.16 (Frobenius)**.** *If S is an involutive distribution then for every $x \in M$ there exists such a local coordinate system $y^1, \dots, y^n$ in its neighbourhood that the vector fields $\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}$ form the basis of the distribution S on U.*

It follows that every k-dimensional submanifold $y^{k+1} = const., \ldots, y^n = const.$ in U is integral manifold. It means that the involutivity implies the integrability. Hence the distribution is integrable if and only if it is involutive.

An extensive proof of the theorem can be found in [8].

3 Trident snake control system

The first part of this chapter is based on [6], [5] and the second on [7], [9].

3.1 Control system

Our next goal is to define a control system of a trident snake robot. To recall its kinematics let us state the system

$$\begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{pmatrix} = \begin{pmatrix} \sin(\theta+\alpha_1+\phi_1) & -\cos(\theta+\alpha_1+\phi_1) & -(1+\cos\phi_1) \\ \sin(\theta+\alpha_2+\phi_2) & -\cos(\theta+\alpha_2+\phi_2) & -(1+\cos\phi_2) \\ \sin(\theta+\alpha_3+\phi_3) & -\cos(\theta+\alpha_3+\phi_3) & -(1+\cos\phi_3) \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix}.$$

Our model in a sense of previous chapter can be transformed into the system of ODEs which describes local controllability. Hence we have the following control system $\dot{q} = Gu$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \sin(\theta+\alpha_1+\phi_1) & -\cos(\theta+\alpha_1+\phi_1) & -1-\cos(\phi_1) \\ \sin(\theta+\alpha_2+\phi_2) & -\cos(\theta+\alpha_2+\phi_2) & -1-\cos(\phi_2) \\ \sin(\theta+\alpha_3+\phi_3) & -\cos(\theta+\alpha_3+\phi_3) & -1-\cos(\phi_3) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

where the vector $(u_1, u_2, u_3) = (\dot{x}, \dot{y}, \dot{\theta}) = \dot{\mathbf{w}}$ is the vector of controlling parameters. Then control matrix G is a 6×3 matrix whose columns are considered as the controlling vector fields g_1, g_2, g_3. Equivalently we have

$$\dot{q} = g_1(q)u_1 + g_2(q)u_2 + g_3(q)u_3.$$

If we eliminate θ from the matrix A (i.e. $\dot{\phi} = A(\phi)R_\theta^T\dot{\mathbf{w}}$), the appropriate control system changes to the following form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \sin(\alpha_1+\phi_1) & -\cos(\alpha_1+\phi_1) & -1-\cos(\phi_1) \\ \sin(\alpha_2+\phi_2) & -\cos(\alpha_2+\phi_2) & -1-\cos(\phi_2) \\ \sin(\alpha_3+\phi_3) & -\cos(\alpha_3+\phi_3) & -1-\cos(\phi_3) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}.$$

Now the computation of Lie brackets for controlling vector fields of our trident snake robot can be realized. For our vector fields g_1 and g_2 with six coordinates it is useful to simplify the computation in a matrix form where Jacobi matrix appears.

$$g_1 = \cos\partial_{x_1} + \sin\partial_{x_2} + \sin(\phi_1 - \frac{2}{3}\pi)\partial_{x_4} + \sin(\phi_2)\partial_{x_5} + \sin(\phi_3 + \frac{2}{3}\pi)\partial_{x_6}$$
$$g_2 = -\sin\partial_{x_1} + \cos\partial_{x_2} - \cos(\phi_1 - \frac{2}{3}\pi)\partial_{x_4} - \cos(\phi_2)\partial_{x_5} - \cos(\phi_3 + \frac{2}{3}\pi)\partial_{x_6}$$

Thus for vector fields g_1, g_2 we have

$$[g_1, g_2] = \frac{\partial g_1}{\partial x^i} g_2 - \frac{\partial g_2}{\partial x^i} g_1 =$$

$$= \begin{pmatrix} \frac{\partial\cos\theta}{\partial x} & \frac{\partial\cos\theta}{\partial y} & . & . & \frac{\partial\cos\theta}{\partial\phi_2} & \frac{\partial\cos\theta}{\partial\phi_3} \\ \frac{\partial\sin\theta}{\partial x} & \frac{\partial\sin\theta}{\partial y} & . & . & \frac{\partial\sin\theta}{\partial\phi_2} & \frac{\partial\sin\theta}{\partial\phi_3} \\ \frac{\partial 0}{\partial x} & \frac{\partial 0}{\partial y} & . & . & \frac{\partial 0}{\partial\phi_2} & \frac{\partial 0}{\partial\phi_3} \\ \frac{\partial(\sin(\phi_1 - \frac{2}{3}\pi))}{\partial x} & \frac{\partial(\sin(\phi_1 - \frac{2}{3}\pi))}{\partial y} & . & . & \frac{\partial(\sin(\phi_1 - \frac{2}{3}\pi))}{\partial\phi_2} & \frac{\partial(\sin(\phi_1 - \frac{2}{3}\pi))}{\partial\phi_3} \\ \frac{\partial(\sin(\phi_2))}{\partial x} & \frac{\partial(\sin(\phi_2))}{\partial y} & . & . & \frac{\partial(\sin(\phi_2))}{\partial\phi_2} & \frac{\partial(\sin(\phi_2))}{\partial\phi_3} \\ \frac{\partial(\sin(\phi_3 + \frac{2}{3}\pi))}{\partial x} & \frac{\partial(\sin(\phi_3 + \frac{2}{3}\pi))}{\partial y} & . & . & \frac{\partial(\sin(\phi_3 + \frac{2}{3}\pi))}{\partial\phi_2} & \frac{\partial(\sin(\phi_3 + \frac{2}{3}\pi))}{\partial\phi_3} \end{pmatrix} \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \\ -\cos(\phi_1 - \frac{2}{3}\pi) \\ -\cos(\phi_2) \\ -\cos(\phi_3 + \frac{2}{3}\pi) \end{pmatrix} -$$

$$- \begin{pmatrix} \frac{\partial(-\sin\theta)}{\partial x} & \frac{\partial(-\sin\theta)}{\partial y} & . & . & \frac{\partial(-\sin\theta)}{\partial\phi_2} & \frac{\partial(-\sin\theta)}{\partial\phi_3} \\ \frac{\partial(\cos\theta)}{\partial x} & \frac{\partial(\cos\theta)}{\partial y} & . & . & \frac{\partial(\cos\theta)}{\partial\phi_2} & \frac{\partial(\cos\theta)}{\partial\phi_3} \\ \frac{\partial 0}{\partial x} & \frac{\partial 0}{\partial y} & . & . & \frac{\partial 0}{\partial\phi_2} & \frac{\partial 0}{\partial\phi_3} \\ \frac{\partial(-\cos(\phi_1 - \frac{2}{3}\pi))}{\partial x} & \frac{\partial(-\cos(\phi_1 - \frac{2}{3}\pi))}{\partial y} & . & . & \frac{\partial(-\cos(\phi_1 - \frac{2}{3}\pi))}{\partial\phi_2} & \frac{\partial(-\cos(\phi_1 - \frac{2}{3}\pi))}{\partial\phi_3} \\ \frac{\partial(-\cos(\phi_2))}{\partial x} & \frac{\partial(-\cos(\phi_2))}{\partial y} & . & . & \frac{\partial(-\cos(\phi_2))}{\partial\phi_2} & \frac{\partial(-\cos(\phi_2))}{\partial\phi_3} \\ \frac{\partial(-\cos(\phi_3 + \frac{2}{3}\pi))}{\partial x} & \frac{\partial(-\cos(\phi_3 + \frac{2}{3}\pi))}{\partial y} & . & . & \frac{\partial(-\cos(\phi_3 + \frac{2}{3}\pi))}{\partial\phi_2} & \frac{\partial(-\cos(\phi_3 + \frac{2}{3}\pi))}{\partial\phi_3} \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \\ \sin(\phi_1 - \frac{2}{3}\pi) \\ \sin(\phi_2) \\ \sin(\phi_3 + \frac{2}{3}\pi) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

In the same way we compute all Lie brackets of vector fields g_1, g_2, g_3. They are of the following form

$$g_{12} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \; g_{13} = \begin{pmatrix} \sin\theta \\ \cos\theta \\ 0 \\ \frac{1}{2} - \cos(\phi_1 - \frac{2}{3}\pi) \\ 1 - \cos(\phi_2) \\ \frac{1}{2} - \cos(\phi_3 + \frac{2}{3}\pi) \end{pmatrix}, \; g_{23} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \\ -\frac{\sqrt{3}}{2} + \sin(\phi_1 - \frac{2}{3}\pi) \\ \sin(\phi_2) \\ \frac{\sqrt{3}}{2} + \sin(\phi_3 - \frac{2}{3}\pi) \end{pmatrix}. \tag{6}$$

3.2 Controllability

We need to discuss the controllability of a given system

$$\dot{q} = \sum_{i=1}^{m} u_i g_i(q), \quad q \in M, \tag{7}$$

where $m = 3$, therefore $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ is a control vector and g_i, $i = 1, 2, 3$, are smooth vector fields on M. In other words, we need to find out if we can join any two points of a system (7) by a trajectory. Or in a similar way if we are able to steer our model to an arbitrary state (i.e. arbitrary point on M) from an arbitrary initial state q_0.

Let us begin with the definition of a reachable set and controllable system.

Definition 3.1. The set $\mathfrak{R}_p$ of points reached by a trajectory of (7) issued from p is called the *reachable set of* $p \in M$.

If the reachable set of any point is equal to the whole manifold M then the system is called *controllable.*

Next step is to define basic notions of this topic. Then we formulate conditions under which our system is controllable.

Let $\Delta = \operatorname{span}\{g_1, \dots, g_m\}$ be the distribution associated with the control system

$$\dot{q} = g_1(q)u_1 + \cdots + g_m(q)u_m, \tag{8}$$

where $m = 3$. We define $\Delta^1 = \Delta$ and

$$\Delta^s = \Delta^{s-1} + [\Delta^1, \Delta^{s-1}],$$

where

$$[\Delta^1, \Delta^{s-1}] = \operatorname{span}\{[g, h] : g \in \Delta^1, h \in \Delta^{s-1}\}.$$

We can see that $\Delta^s \subset \Delta^{s+1}$. The chain of the distributions Δ^s is defined as a *filtration* associated with the distribution $\Delta = \Delta^1$. Each Δ^s is defined to be spanned by the controlling vector fields plus the vector fields formed by taking up to $s - 1$ Lie brackets of the generators, i.e., elements of Δ^1.

The filtration for our trident snake robot is in the following form

$$\begin{aligned}
\Delta^1 =&\operatorname{span}\{g_1, g_2, g_3\}\\
\Delta^2 =&\Delta^1 + [\Delta^1, \Delta^1] = \operatorname{span}\{g_1, g_2, g_3\} + [\operatorname{span}\{g_1, g_2, g_3\}, \operatorname{span}\{g_1, g_2, g_3\}] = \\
=&\operatorname{span}\{g_1, g_2, g_3, [g_1, g_2], [g_1, g_3], [g_2, g_3]\}.
\end{aligned}$$

Hence we see that the filtration is $(3, 6)$.

Let us introduce a property of a filtration to define a property of our control system. We say that a filtration is *regular* in a neighbourhood U of q_0 if

$$\operatorname{rank} \Delta^s(q) = \operatorname{rank} \Delta^s(q_0) \qquad \forall q \in U.$$

We say that the control system (8) is *regular* if the corresponding filtration is regular.

As we mentioned in the previous part, the set of smooth vector fields on the manifold M with the Lie bracket is a Lie algebra. We denote it by $\chi(M)$.

Then *Lie algebra generated by* $g_1, \dots, g_m$ is defined to be $Lie(g_1, \dots, g_m) = \bigcup_{s \geq 1} \Delta^s$. Which means that for a trident snake robot the appropriate Lie algebra is $Lie(g_1, g_2, g_3) = \text{span}\{g_1, g_2, g_3, g_{12}, g_{23}, g_{13}\}$.

An important step to define controlability of a system is Chow's condition. Thus we say that a system (7) (or vector fields $g_1, \dots, g_m$) satisfies *Chow's condition* if

$$Lie(g_1, \dots, g_m)(q) = T_q M, \quad \forall q \in M.$$

In other words, for any $q \in M$, there exists an integer $r = r(q)$ such that $\dim \Delta^r(q) = n$ (where n is the dimension of the manifold M). Hence for trident snake $r = 2$ because $\dim \Delta^2 = 6$ and the dimension of a manifold M is also 6.

Theorem 3.2 (Chow-Rashevsky)**.** *If M is connected and if* (7) *satisfies Chow's condition, then any two points of M can be joined by a trajectory of* (7)*.*

The proof of this theorem can be found in several publications, for example in [7].

For us the main idea of the theorem can be found in the fact that the dimension of the controlling distribution of the trident snake robot needs to have the same dimension as the state space manifold M. In our case $\dim M$=6 and we know that the distribution Δ^2 has exactly the required dimension. Hence the Lie algebra generated by three vector fields g_1, g_2, g_3 with their Lie brackets g_{12}, g_{13}, g_{23} corresponds to our controlling distribution of the trident snake robot which is due to this fact locally controllable.

Note also that the controllability is local due to the fact that we assumed only linear system (7) (in u) which corresponds locally to the general (nonlinear) one.

4 Geometry of Nonholonomic Systems

In this section we use [7],[5],[9] and [10].

The aim is to discuss the linearization of a control system. Usually, linearization of a dynamical system is reached by the theory of equilibrium points and Jacobi matrices. For more details look for example at [10]. We recall basic definitions and results for nonholonomic systems which are discussed in [7].

4.1 Nonholonomic systems

In general, the main property of a nonholonomic system is that it cannot move in arbitrary direction in its configuration space. For our trident snake model it is caused by passive wheels which are assumed not to slip, nor slide sideways. But let us define the nonholonomic system properly as we can find in [7].

Let us consider a nonlinear control system in $\mathbb{R}^n$,

$$\dot{x} = f(x, u), \tag{9}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^n$ is the control. Given a control law $u(t),\ t \in [0, T]$, a trajectory associated with $u(\cdot)$ is defined as a solution of the nonautonomous ordinary differential equation $\dot{x} = f(x, u(t))$.

Next step is to analyze the system. We want to find a solution $u(\cdot)$ of (9) as a function $u(t) = k(x(t))$ where the differential equation $\dot{x} = f(x, k(x))$ is stable.

To find such a system we can use locally first-order approximation of the original system.

We assume that the control system depends linearly on u, that is,

$$\dot{x} = \sum_{i=1}^{m} u_i X_i(x). \tag{10}$$

Thus we can take a linearization of this system for every equilibrium point $(x_0, 0)$. For the linearized system, the reachable set from a point x is the affine subset $x + \Delta(x_0)$, where $\Delta(x_0) = \mathrm{span}\{X_1(x_0), \ldots, X_m(x_0)\}$. Thus the reachable set from an arbitrary point $x \in \mathbb{R}^n$ may be restricted by some conditions. In other words, we distiguish two cases depending on the dimension of $\Delta(x_0)$:

- if $\dim\Delta(x_0) = n$, then the linearized system is controllable.
- if $\dim\Delta(x_0) < n$, then the linearized system is not controllable. However system may be controllable as we will see later in this text.

Now we can say that nonholonomic systems are the systems which belong to the second category and have a form (10). It means that these systems are limited by conditions which cause that the reachable set of an arbitrary point $x \in \mathbb{R}^n$ is not the whole state space. In other words, the system cannot move to arbitrary state due to the nonholonomic constraints.

From previous text we know that our model of tridet snake is also described by nonholonomic system but it is controllable due to Chow-Rashevsky theorem. However to simplify this system we need to study their properties.

Usually the linearization is a first-order approximation with respect to a Euclidean (Riemannian) distance. However for nonholonomic systems the underlying distance is a sub-Riemannian one and it behaves very differently from a Euclidean one. Therefore the local behaviour of nonholonomic system should be understood through the study of a first-order approximation with respect to the sub–Riemannian distance, not through the linearized system.

So let introduce now the definition of nonholonomic system and sub–Riemannian distance.

Definition 4.1. A *nonholonomic system* on M is a control system which is of the form

$$\dot{q} = u_1 X_1(q) + \cdots + u_m X_m(q), \quad q \in M, \quad u = (u_1, \ldots, u_m) \in \mathbb{R}^m, \tag{11}$$

where $m > 1$ is an integer and $X_1, \ldots, X_m$ are smooth vector fields on M.

The system (11) determines a family of vector spaces,

$$\Delta(q) = \mathrm{span}\{X_1(q), \ldots, X_m(q)\} \subset T_q M, \quad q \in M. \tag{12}$$

As we can see, the dimension of $\Delta(q)$ is a function of q, thus in every point the dimension can be different. However if it is constant, then Δ defines a distribution on M.

Now let recall some basic information about distributions. The distribution assigns a subspace of the tangent space to each point in $\mathbb{R}^n$ in a smooth way. A special case is a distribution defined by a set of smooth vector fields $g_1, \ldots, g_m$. In this case we define the distribution as

$$\Delta = \mathrm{span}\{g_1, \ldots, g_m\},$$

where we take the span over the set of smooth real-valued functions on M.

Evaluated at any point $q \in M$, the distribution defines a linear subspace of the tangent space

$$\Delta(q) = \operatorname{span}\{g_1(q), \dots, g)_m(q)\} \subset T_q M.$$

The distribution is said to be *regular* if the dimension of the subspace $\Delta(q)$ does not vary with q.

Next step is to define a sub–Riemannian metric but for that we need some prerequisites.

Definition 4.2. A trajectory of system (11) is a path $\gamma : [0, T] \to M$ for which there exists a function $u(\cdot) \in L^1([0, T], \mathbb{R}^m)$ such that γ is a solution of the ordinary differential equation,

$$\dot{q}(t) = \sum_{i=1}^{m} u_i(t) X_i(q(t)), \quad \text{for a.e. } t \in [0, T]. \tag{13}$$

Such a function $u(\cdot)$ is called a *control* associated with γ.

We can say that every trajectory is an absolutely continuous path γ on M such that $\dot{\gamma} \in \Delta(\gamma(t))$ for almost every $t \in [0, T]$.

Definition 4.3. The function $g : TM \to \langle 0, \infty)$ given by

$$g(q, v) = \inf \left\{ u_1^2 + \dots + u_m^2 : \sum_{i=1}^{m} u_i X_i(q) = v \right\}, \tag{14}$$

for $q \in M$ and $v \in T_q M$, where we take that $\inf \emptyset = +\infty$ and satisfies:

- if $v \notin \Delta(q)$, then $g(q, v) = +\infty$;
- if $v \in \Delta(q)$, then the infimum is attained at a unique value $u^* \in \mathbb{R}^m$, and thus $g(q, v) = \|u^*\|^2$ where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^m$

is called the *sub-Riemannian metric.*

If we have a new metric we can define a distance in a similar way as in Riemannian geometry.

Definition 4.4. The *length* of an absolutely continuous path $\gamma(t),\ t \in [0, T]$, is

$$l(\gamma) = \int_0^T \sqrt{g(\gamma(t), \dot{\gamma}(t))} dt.$$

The *sub-Riemannian distance* on M associated with the nonholonomic system (11) is defined by

$$d(p, q) = \inf l(\gamma), \tag{15}$$

where the infimum is taken over all absolutely continuous paths γ joining p and q.

It is important to mention that the length of the path is independent of the parametrization of the path. Therefore we also understand the sub–Riemannian distance $d(p, q)$ as a minimal time needed for the nonholonomic system to go from p to q with bounded controls. Thus it means

$$d(p, q) = \inf \begin{Bmatrix} T \geq 0 : \exists \text{ a trajectory } \gamma_u : [0, T] \to M \quad \text{s.t.} \\ \gamma_u(0) = p, \ \gamma_u(T) = q, \\ \text{and } \|u(t)\| \leq 1 \quad \text{for a.e. } t \in [0, T] \end{Bmatrix}$$

Hence we can understand a first–order approximations with respect to the time of nonholonomic systems as a first–order approximations with respect to the sub–Riemannian distance.

4.2 Nonholonomic orders

After defining a new distance we can continue in a process of simplifying the nonholonomic system.

We have a nonholonomic system $\dot{q} = \sum_{i=1}^{m} u_i X_i(q)$ on a manifold M satisfying Chow‘s condition and the induced sub-Riemannian distance d. The local behaviour of this system should be described by an approximation to the fisrt–order with respect to d.

In the whole section we work with local objects. Hence, throughout the section we fix a point $p \in M$ and an open neighbourhood U of p that we identify with a neighbourhood of 0 in $\mathbb{R}^n$ through some local coordinates.

Let us begin with the basic notions of this topic which are necessary to define the fist–order approximation of the system. The first is a nonholonomic order of a function and a vector field at a point.

Definition 4.5. Let $f : M \to \mathbb{R}$ be a continuous function. The *nonholonomic order of f at p*, denoted by $\operatorname{ord}_p(f)$, is the real number defined by

$$\operatorname{ord}_p(f) = \sup\{s \in \mathbb{R} : f(q) = O(d(p, q)^s)\}.$$

This order is always nonnegative, moreover $\mathrm{ord}_p(f) = 0$ if $f(p) \neq 0$ and $\mathrm{ord}_p(f) = +\infty$ if $f(p) = 0$.

We can also define it in another way using Lie derivatives. Therefore let us recall them.

Nonholonomic derivatives of order 1 of f are the Lie derivatives $X_1 f, \ldots, X_m f$. We define this derivative in the following way. Let X be a smooth vector field and $f \in C^\infty(M)$ a smooth function on M. The *Lie derivative* of f with respect to X is a new function $Xf : M \to \mathbb{R}$ defined by

$$Xf(p) = X_p f.$$

In coordinate chart (U, φ), if we write $X = \sum_{i=1}^n X_i(x) \frac{\partial}{\partial x_i}$, then

$$Xf(x) = \sum_{i=1}^n \frac{\partial f}{\partial x_i} X_i(x),$$

where all partial derivatives are evaluated at $x = \varphi(p)$.

Therefore next way how to define the nonholonomic order of a smooth function is given by the formula

$$\mathrm{ord}_p(f) = \min\{s \in \mathbb{N} : \exists i_1, \ldots, i_s \in \{1, \ldots, m\} \text{ s.t. } (X_{i_1}, \ldots, X_{i_s} f)(p) \neq 0\},$$

where we adopt the convention that $\min \emptyset = +\infty$.
In other words, we differentiate function f until the expression is not equal 0 in p.

Example For better understanding let us compute a short example in a Euclidean space. We have $M = \mathbb{R}^3$ with three coordinates x, y, z, therefore the sub–Riemannian metric is simply the Euclidean metric on $\mathbb{R}^3$. Hence we know that for this case the nonholonomic orders correspond to the standard ones.

For example the nonholonomic order of a function $f = 3x + \frac{x^2}{2} + y^2 - 6y^3 - z$ at $p = 0$ is $\mathrm{ord}_p(f) = 1$ because we need to differentiate once the function by x to get a nonzero expression. If the function had a constant part then the order would be 0 which is the minimal value of this notion because it cannot be negative.

We can extend the notion of nonholonomic order to vector fields in the following way.

Definition 4.6. Let X be a smooth vector field at p. The *nonholonomic order of X at p*, denoted by $\mathrm{ord}_p(X)$, is the real number defined by

$$\mathrm{ord}_p(X) = \sup\{\sigma \in \mathbb{R} : \mathrm{ord}_p(Xf) \geq \sigma + \mathrm{ord}_p(f), \quad \forall f \in C^\infty(p)\}.$$

The order of a differential operator is defined in the same way.
Note that $\mathrm{ord}_p(X) \in \mathbb{Z}$ since the order of a smooth function is an integer.

Example Let us show an example to understand the meaning clearly.

An important remark for this topic is that the differential operator in a Euclidean space is the negative of its usual order. It means that for example ∂_{x_i} is of order -1. We will use this fact in the following computation.

Let us compute a nonholonomic order of specific vector fields. Consider vector fields X_1, X_2 on $\mathbb{R}^3$ (with coordinates x, y, z) of the following form

$$\begin{aligned} X_1 &= \partial_x + \frac{y}{3}\partial_z, \\ X_2 &= \partial_y + \frac{x}{3}\partial_z. \end{aligned}$$

Then the computation leads to this result.

$$\begin{aligned} X_1 x(0) &= 1 \\ X_2 y(0) &= 1 \\ X_1 z(0) &= 0 \\ X_2 z(0) &= 0 \\ X_1 X_2 z(0) &= \frac{1}{3}. \end{aligned}$$

Hence the order of a coordinate functions x and y at 0 is 1 and at z it is 2. Then the orders of our vector fields are $\mathrm{ord}_0 X_1 = \sigma_1 = -1$ and $\mathrm{ord}_0 X_2 = \sigma_2 = -1$ since

$$\begin{aligned} \mathrm{ord}_0(X_1 x) &= 0 \geq \sigma_1 + \mathrm{ord}_0 x, \\ \mathrm{ord}_0(X_1 y) &= 0 \geq \sigma_1 + \mathrm{ord}_0 y, \\ \mathrm{ord}_0(X_1 z) &= 1 \geq \sigma_1 + \mathrm{ord}_0 z, \\ \mathrm{ord}_0(X_2 x) &= 0 \geq \sigma_2 + \mathrm{ord}_0 x, \\ \mathrm{ord}_0(X_2 y) &= 0 \geq \sigma_2 + \mathrm{ord}_0 y, \\ \mathrm{ord}_0(X_2 z) &= 1 \geq \sigma_2 + \mathrm{ord}_0 z. \end{aligned}$$

Next step is to define the first–order approximation of a family of vector fields near some point in a sense of orders.

Definition 4.7. A family of m vector fields $(\hat{X}_1, \dots, \hat{X}_m)$ defined near p is called a *first–order approximation of* $(X_1, \dots, X_m)$ *at* p if the vector fields $X_i - \hat{X}_i,\ i = 1, \dots, m$, are of order ≥ 0 at p.

Note that in the following text we will see that approximations to the first–order appear as nilpotent approximations. For us it means that $X_1, \dots, X_m$ are approximated by vector fields that generate a nilpotent Lie algebra.

Now we know the meaning of a first–order approximation in the sense of a sub–Riemannian metric. But the computation leads to some new notions. For example we need to set a suitable system of coordinates.

4.3 Privileged coordinates

In the previous section we have introduced the sets of vector fields Δ^s, defined by $\Delta^s = \text{span}\{X_I : |I| \leq s\}$.

Because vector fields $X_1, \dots, X_m$ satisfy Chow's condition we know that

$$\Delta^1(p) \subset \Delta^2(p) \subset \cdots \subset \Delta^{r-1}(p) \subsetneq \Delta^r(p) = T_pM, \tag{16}$$

where $r = r(p)$ is the degree of nonholonomy at p.
The *growth vector at* p is the r-tuple of integers $(n_1(p), \dots, n_r(p))$, where $n_s(p) = \dim \Delta^s(p)$. The first integer in the growth vector is the rank $n_1(p) \leq m$ of the family $X_1(p), \dots, X_m(p)$ and the last one $n_r(p) = n$ is the dimension of the manifold M.

In the case of trident snake robot we have the following chain

$$\Delta^1(p) = \text{span}\{g_1(p), g_2(p), g_3(p)\} \subset \Delta^2 = \text{span}\{g_1(p), g_2(p), g_3(p), g_{12}(p), g_{13}(p), g_{23}(p)\} = T_pM.$$

Due to the fact that $\dim\Delta^1(p) = n_1(p) = 3$ and $\dim\Delta^2(p) = n_2(p) = 6$ the growth vector at p of trident snake is $n(p) = (3, 6)$.

To simplify the notation we denote Δ^s as the map $q \mapsto \Delta^s(q)$. This map is distribution if and only if $n_s(q)$ is constant on M. If the growth vector is constant in a neighbourhood of p then the point p is called a *regular point* (w.r.t. $X_1, \dots, X_m$). Otherwise, p is a *singular point*. Hence near a regular point all maps Δ^s are locally distributions.

We also define the *weights at* p as $w_i = w_i(p),\ i = 1, \dots, n$ where $w_j = s$ if

$n_{s-1}(p) < j \leq n_s(p)$ and $n_0 = 0$. Therefore we have

$$\begin{aligned}
&w_1 = \cdots = w_{n_1} = 1 \\
&w_{n_1+1} = \cdots = w_{n_2} = 2 \\
&\quad \cdots \\
&\quad \cdots \\
&w_{n_{r-1}+1} = \cdots = w_{n_r} = r.
\end{aligned}$$

Now let us show the weights at p of a trident snake robot with a growth vector $n = (3, 6)$, in other words $n_1(p) = 3$ and $n_2(p) = 6$. We start with $n_0 = 0$ and thus we get

$$\begin{aligned}
w_1 &= 1, &&\text{because} \quad n_0(p) = 0 < 1 \leq 3 = n_1(p), \\
w_2 &= 1, &&\text{because} \quad n_0(p) = 0 < 2 \leq 3 = n_1(p), \\
w_3 &= 1, &&\text{because} \quad n_0(p) = 0 < 3 \leq 3 = n_1(p), \\
w_4 &= 2, &&\text{because} \quad n_1(p) = 3 < 4 \leq 6 = n_2(p), \\
w_5 &= 2, &&\text{because} \quad n_1(p) = 3 < 5 \leq 6 = n_2(p), \\
w_6 &= 2, &&\text{because} \quad n_1(p) = 3 < 6 \leq 6 = n_2(p).
\end{aligned}$$

It leads to a nondecreasing sequence of weights at p $w_1(p) \leq w_2(p) \leq w_3(p) \leq w_4(p) \leq w_5(p) \leq w_6(p)$.

Now we have all necessary notions to define privileged coordinates.

Definition 4.8. A *system of privileged coordinates at p* is a system of local coordinates $(z_1, \ldots, z_n)$ such that $\mathrm{ord}_p(z_j) = w_j$ for $j = 1, \ldots, n$.

Thus if we consider a trident snake robot, the system of privileged coordinates at p has to satisfy the following conditions

$$\begin{aligned}
\mathrm{ord}_p(z_1) &= w_1 = 1, \\
\mathrm{ord}_p(z_2) &= w_2 = 1, \\
\mathrm{ord}_p(z_3) &= 1, \\
\mathrm{ord}_p(z_4) &= 2, \\
\mathrm{ord}_p(z_5) &= 2, \\
\mathrm{ord}_p(z_6) &= 2.
\end{aligned}$$

Privileged coordinates can be used to compute orders. This is the reason why we introduce them, to compute easily the orders to get the first–order approximation of the vector fields.

We fix a system of privileged coordinates $(z_1, \dots, z_n)$ at p. Given a sequence of integers

$$\alpha = (\alpha_1, \dots, \alpha_n),$$

we define the weighted degree of the monomial

$$z^\alpha = z^{\alpha_1} \dots z^{\alpha_n}$$

to be

$$w(\alpha) = w_1\alpha_1 + \dots + w_n\alpha_n.$$

Let us show the meaning of this notion in a short example. Consider the 3–dimensional case (z_1, z_2, z_3) with coordinate weights $(1, 1, 2)$. Then the weighted degree of the monomial $z^\alpha = z_1^2 z_2 z_3^2$ is equal to $w(\alpha) = 2 \cdot 1 + 1 \cdot 1 + 2 \cdot 2 = 7$.

The weighted degree of the monomial vector field $z^\alpha \partial_{z_j}$ is defined as $w(\alpha) - w_j$.

Proposition 4.9. *For a smooth function f with a Taylor expansion*

$$f(z) \sim \sum_\alpha c_\alpha z^\alpha,$$

the order of f is the least weighted degree of monomials having a nonzero coefficient in the Taylor series.

Example For better understanding let us again show an example. Consider a smooth function $f(z_1, z_2, z_3) = z_1^3 + z_1 z_2 + z_2^3 + z_1 z_2 z_3 + z_3$ with coordinate weights $(1, 1, 2)$. Because this function is already in the form of Taylor expansion we can immediately compute a weighted degree of each monomial in the following way

$$\begin{aligned}
z_1^3 = z_1^3 z_2^0 z_3^0 &\Rightarrow \alpha = (3, 0, 0), \quad w(\alpha) = 1 \cdot 3 + 1 \cdot 0 + 2 \cdot 0 = 3 \\
z_1 z_2 = z_1^1 z_2^1 z_3^0 &\Rightarrow \alpha = (1, 1, 0), \quad w(\alpha) = 1 \cdot 1 + 1 \cdot 1 + 2 \cdot 0 = 2 \\
z_2^3 = z_1^0 z_2^3 z_3^0 &\Rightarrow \alpha = (0, 3, 0), \quad w(\alpha) = 1 \cdot 0 + 1 \cdot 3 + 2 \cdot 0 = 3 \\
z_1 z_2 z_3 = z_1^1 z_2^1 z_3^1 &\Rightarrow \alpha = (1, 1, 1), \quad w(\alpha) = 1 \cdot 1 + 1 \cdot 1 + 2 \cdot 1 = 4 \\
z_3 = z_1^0 z_2^0 z_3^1 &\Rightarrow \alpha = (0, 0, 1), \quad w(\alpha) = 1 \cdot 0 + 1 \cdot 0 + 2 \cdot 1 = 2.
\end{aligned}$$

The order of f is defined as the least weighted degree of monomials having a nonzero coefficient therefore for our function f it is equal to 2.

Similarly we introduce an order of a vector field defined by the weighted degree of a monomial vector fields.

Proposition 4.10. *For a vector field X with a Taylor expansion*

$$X(z) \sim \sum_{\alpha} a_{\alpha,j} z^{\alpha} \partial_{z_j},$$

the order of X is the least weighted degree of a monomial vector fields having a nonzero coefficient in the Taylor series.

The proof of both previous statements can be found in [7].

Bellaïche's algorithm The construction of privileged coordinates can be realized by Bellaïche's algorithm. But other constructions exsist. Due to the fact that in our trident snake model the weights at p are 1 and 2 we use the first two steps of this algorithm only. Therefore it may be presented in the following way:

1. Choose an adapted frame $Y_1, \ldots, Y_n$ at p.
2. Choose coordinates $(y_1, \ldots, y_n)$ centered at p such that $\partial_{y_i}|_p = Y_i(p)$.

We get linearly adapted coordinates $(y_1, \ldots, y_n)$ from which we can obtain the privileged ones which have the form

$$\begin{aligned}
z_1 &= y_1, \\
z_2 &= y_2 + \mathrm{pol}(y_1), \\
&\ldots \\
&\ldots \\
z_n &= y_n + \mathrm{pol}(y_1, \ldots, y_{n-1}),
\end{aligned}$$

where pol stands for a polynomial function with neither constant nor linear terms.

Let us continue with a computation of an adapted frame for which the condition $\partial_{z_i}|_p = Y_i(p)$ holds in the particular example.

Example Let us consider a 2–dimensional manifold M with the coordinate functions denoted by y_1, y_2. Furthermore, let us denote the basis of a vector space T_pM by $(\partial_{y_1}, \partial_{y_2}), p \in M$. If we consider two vector fields

$$\begin{aligned}
Y_1 &= \cos\theta \partial_{y_1} + \sin\theta \partial_{y_2}, \\
Y_2 &= \sin\theta \partial_{y_1} + \cos\theta \partial_{y_2}, \quad \theta \in \mathbb{R},
\end{aligned}$$

the question is what is the exact form of a coordinate transformation $y := (y_1, y_2) \rightsquigarrow (z_1, z_2) =: z$ such that the condition

$$\frac{\partial}{\partial z_i} \mid_p = Y_i \mid_p, \quad i = 1, 2, \tag{17}$$

holds in $p \in M$. Let us denote by $[Y_k^i]_y$ the i–th coordinate of a vector Y_k in the basis y. In our case, $i, k \in \{1, 2\}$ then we have

$$\begin{aligned} [Y_1^1]_y &= \cos\theta & [Y_1^2]_y &= \sin\theta \\ [Y_2^1]_y &= \sin\theta & [Y_2^2]_y &= \cos\theta \end{aligned}$$

Clearly, the condition (17) reads

$$[Y_1]_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } [Y_2]_z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The transformation law for vector fields under the coordinate change $y \rightsquigarrow z$ reads

$$\left[Y_k^i\right]_z = \frac{\partial z_i}{\partial y_j} \left[Y_k^j\right]_y$$

for $i, k \in \{1, 2\}$ and concerning the Einstein summation convention, i.e. summing over j ranging from 1 to 2. Particularly, in the matrix form we have

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = [Y_1]_z = \begin{pmatrix} \frac{\partial z_1}{\partial y_1} \left[Y_1^1\right]_y + \frac{\partial z_1}{\partial y_2} \left[Y_1^2\right]_y \\ \frac{\partial z_2}{\partial y_1} \left[Y_1^1\right]_y + \frac{\partial z_2}{\partial y_2} \left[Y_1^2\right]_y \end{pmatrix}$$

and

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = [Y_2]_z = \begin{pmatrix} \frac{\partial z_1}{\partial y_1} \left[Y_2^1\right]_y + \frac{\partial z_1}{\partial y_2} \left[Y_2^2\right]_y \\ \frac{\partial z_2}{\partial y_1} \left[Y_2^1\right]_y + \frac{\partial z_2}{\partial y_2} \left[Y_2^2\right]_y \end{pmatrix}.$$

Thus we get a system of PDEs

$$\frac{\partial z_1}{\partial y_1} \cos\theta + \frac{\partial z_1}{\partial y_2} \sin\theta = 1, \tag{18}$$

$$\frac{\partial z_2}{\partial y_1} \cos\theta + \frac{\partial z_2}{\partial y_2} \sin\theta = 0, \tag{19}$$

$$\frac{\partial z_1}{\partial y_1} \sin\theta + \frac{\partial z_1}{\partial y_2} \cos\theta = 0, \tag{20}$$

$$\frac{\partial z_2}{\partial y_1} \sin\theta + \frac{\partial z_2}{\partial y_2} \cos\theta = 1. \tag{21}$$

The couple of equations (18) and (20) form a system of linear equations of the form

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial z_1}{\partial y_1} \\ \frac{\partial z_1}{\partial y_2} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

with the solution

$$\begin{pmatrix} \frac{\partial z_1}{\partial y_1} \\ \frac{\partial z_1}{\partial y_2} \end{pmatrix} = \frac{1}{\cos 2\theta} \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\cos 2\theta} \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}. \tag{22}$$

The first row leads to

$$\frac{\partial z_1}{\partial y_1} = \frac{1}{\cos 2\theta} \cos\theta$$

which gives us by integration that

$$z_1 = \frac{1}{\cos 2\theta} \cos\theta y_1 + c(y_1) \tag{23}$$

and consequently, the second row reads that

$$\frac{\partial z_1}{\partial y_2} = \frac{1}{\cos 2\theta}(-\sin\theta),$$

which together with (23) leads to

$$z_1 = \frac{1}{\cos 2\theta} \cos\theta y_1 - \frac{1}{\cos 2\theta} \sin\theta y_2.$$

Similarly, equations (19) and (21) imply that

$$z_2 = -\frac{1}{\cos 2\theta} \sin\theta y_1 + \frac{1}{\cos 2\theta} \cos\theta y_2,$$

i.e. we obtained the exact form of the transformation equations in the matrix form

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{1}{\cos 2\theta} \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

The same computation can be done for a trident snake robot. We use a software Maple to compute a new basis. We present the Maple code using Differential geometry and Lie Algebras packages which leads to the following result.

The computation starts with creating a state space.

> $with(DifferentialGeometry) : with(LieAlgebras) :$

> $DGsetup([x_1, x_2, x_3, x_4, x_5, x_6], M, verbose)$

The following coordinates have been protected:

$$[x_1, x_2, x_3, x_4, x_5, x_6]$$

The following vector fields have been defined and protected:

$$[D\_x_1, D\_x_2, D\_x_3, D\_x_4, D\_x_5, D\_x_6]$$

The following differential 1-forms have been defined and protected:

$$[dx_1, dx_2, dx_3, dx_4, dx_5, dx_6]$$

frame name: M **(1)**

We define a basis of our space:

M > $B := [D\_x_1, D\_x_2, D\_x_3, D\_x_4, D\_x_5, D\_x_6]$

$$B := [D\_x_1, D\_x_2, D\_x_3, D\_x_4, D\_x_5, D\_x_6] \tag{2}$$

M > $\alpha_1 := -\frac{2\cdot\pi}{3} : \alpha_2 := 0 : \alpha_3 := \frac{2\cdot\pi}{3} :$

Here we have a list of vector fields and their Lie brackects at q=(0,0,0,0,0,0):

M > $X_1 := evalDG(D\_x_1 + \sin(x_4 + \alpha_1)D\_x_4 + \sin(x_5 + \alpha_2)D\_x_5 + \sin(x_6 + \alpha_3)D\_x_6)$

$$X_1 := D\_x_1 - \sin\left(x_4 + \frac{1}{3}\pi\right) D\_x_4 + \sin(x_5)\, D\_x_5 + \cos\left(x_6 + \frac{1}{6}\pi\right) D\_x_6 \tag{3}$$

M > $X_2 := evalDG(D\_x_2 - \cos(x_4 + \alpha_1)D\_x_4 - \cos(x_5 + \alpha_2)D\_x_5 - \cos(x_6 + \alpha_3)D\_x_6)$

$$X_2 := D\_x_2 + \cos\left(x_4 + \frac{1}{3}\pi\right) D\_x_4 - \cos(x_5)\, D\_x_5 + \sin\left(x_6 + \frac{1}{6}\pi\right) D\_x_6 \tag{4}$$

M > $X_3 := evalDG(D\_x_3 - (1 + \cos(x_4))D\_x_4 - (1 + \cos(x_5))D\_x_5 - (1 + \cos(x_6))D\_x_6)$

$$X_3 := D\_x_3 - (1 + \cos(x_4))\, D\_x_4 - (1 + \cos(x_5))\, D\_x_5 - (1 + \cos(x_6))\, D\_x_6 \tag{5}$$

M > $X_{1, 0} := eval(X_1, [x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 0, x_6 = 0])$

$$X_{1, 0} := D\_x_1 - \frac{1}{2}\sqrt{3}\, D\_x_4 + 0\, D\_x_5 + \frac{1}{2}\sqrt{3}\, D\_x_6 \tag{6}$$

M > $X_{2, 0} := eval(X_2, [x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 0, x_6 = 0])$

$$X_{2, 0} := D\_x_2 + \frac{1}{2}\, D\_x_4 - D\_x_5 + \frac{1}{2}\, D\_x_6 \tag{7}$$

M > $X_{3, 0} := eval(X_3, [x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 0, x_6 = 0])$

$$X_{3, 0} := D\_x_3 - 2\, D\_x_4 - 2\, D\_x_5 - 2\, D\_x_6 \tag{8}$$

M > $X_{12, 0} := eval(LieBracket(X_1, X_2), [x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 0, x_6 = 0])$

$$X_{12, 0} := D\_x_4 + D\_x_5 + D\_x_6 \tag{9}$$

M > $X_{13, 0} := eval(LieBracket(X_1, X_3), [x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 0, x_6 = 0])$

$$X_{13, 0} := -D\_x_4 + 2\, D\_x_5 - D\_x_6 \tag{10}$$

M > $X_{23,\,0} := eval\big(LieBracket(X_2, X_3), [x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 0, x_6 = 0]\big)$

$$X_{23,\,0} := -\sqrt{3}\; D\_x_4 + 0\; D\_x_5 + \sqrt{3}\; D\_x_6 \quad \textbf{(11)}$$

Then we get the coefficients of the vector fields in our basis:

M > $X_{1,\,c} := GetComponents(X_{1,\,0}, B)$

$$X_{1,\,c} := \left[1, 0, 0, -\frac{1}{2}\sqrt{3}, 0, \frac{1}{2}\sqrt{3}\right] \quad \textbf{(12)}$$

M > $X_{2,\,c} := GetComponents(X_{2,\,0}, B)$

$$X_{2,\,c} := \left[0, 1, 0, \frac{1}{2}, -1, \frac{1}{2}\right] \quad \textbf{(13)}$$

M > $X_{3,\,c} := GetComponents(X_{3,\,0}, B)$

$$X_{3,\,c} := [0, 0, 1, -2, -2, -2] \quad \textbf{(14)}$$

M > $X_{12,\,c} := GetComponents(X_{12,\,0}, B)$

$$X_{12,\,c} := [0, 0, 0, 1, 1, 1] \quad \textbf{(15)}$$

M > $X_{13,\,c} := GetComponents(X_{13,\,0}, B)$

$$X_{13,\,c} := [0, 0, 0, -1, 2, -1] \quad \textbf{(16)}$$

M > $X_{23,\,c} := GetComponents(X_{23,\,0}, B)$

$$X_{23,\,c} := [0, 0, 0, -\sqrt{3}, 0, \sqrt{3}] \quad \textbf{(17)}$$

Hence the new basis has the following form due to the fact that the new basis have to correspond with the direction of the original vector fields.

M > $Bn := [X_{1,\,0}, X_{2,\,0}, X_{3,\,0}, X_{12,\,0}, X_{13,\,0}, X_{23,\,0}]$

$$Bn := \Big[D\_x_1 - \frac{1}{2}\sqrt{3}\; D\_x_4 + 0\; D\_x_5 + \frac{1}{2}\sqrt{3}\; D\_x_6, D\_x_2 + \frac{1}{2}\; D\_x_4 - D\_x_5 + \frac{1}{2}\; D\_x_6, D\_x_3 - 2\; D\_x_4 - 2\; D\_x_5 - 2\; D\_x_6, D\_x_4 + D\_x_5 + D\_x_6, -D\_x_4 + 2\; D\_x_5 - D\_x_6, -\sqrt{3}\; D\_x_4 + 0\; D\_x_5 + \sqrt{3}\; D\_x_6\Big] \quad \textbf{(18)}$$

Then a computation of a transformation is computed in the following way.

We start from vector fields where the rotation θ is excluded

$$
\begin{aligned}
g_1 &= \partial_{x_1} + \sin\left(x_4 - \frac{2}{3}\pi\right)\partial_{x_4} + \sin(x_5)\partial_{x_5} + \sin\left(x_6 + \frac{2}{3}\pi\right)\partial_{x_6},\\
g_2 &= \partial_{x_2} - \cos\left(x_4 - \frac{2}{3}\pi\right)\partial_{x_4} - \cos(x_5)\partial_{x_5} - \cos\left(x_6 + \frac{2}{3}\pi\right)\partial_{x_6},\\
g_3 &= \partial_{x_3} - (1+\cos x_4)\partial_{x_4} - (1+\cos x_5)\partial_{x_5} - (1+\cos x_6)\partial_{x_6}.
\end{aligned}
$$

First let us compute Lie brackets of our vector fields in $q = (x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 0, 0, 0, 0)$. The computation leads to the result:

$$
\begin{aligned}
g_4 &= [g_1, g_2] = \partial_{x_4} + \partial_{x_5} + \partial_{x_6},\\
g_5 &= [g_1, g_3] = \partial_{x_4} - 2\partial_{x_5} + \partial_{x_6},\\
g_6 &= [g_2, g_3] = -\sqrt{3}\partial_{x_4} + \sqrt{3}\partial_{x_6}.
\end{aligned}
$$

If we denote by $[g_k^i]_x$ the i–th coordinate of a vector field g_k in the basis x $(i, k = 1, \ldots, 6)$ then we have for the first one:

$$
\begin{aligned}
\left[g_1^1\right]_x &= 1,\\
\left[g_1^2\right]_x &= 0,\\
\left[g_1^3\right]_x &= 0,\\
\left[g_1^4\right]_x &= \sin\left(x_4 - \frac{2}{3}\pi\right),\\
\left[g_1^5\right]_x &= \sin\left(x_5\right),\\
\left[g_1^6\right]_x &= \sin\left(x_6 + \frac{2}{3}\pi\right).
\end{aligned}
$$

The following condition holds in $p \in M$

$$
\frac{\partial}{\partial_{y_i}}|_p = g_i|_p, \qquad i = 1, 2, \ldots, 6. \tag{24}
$$

Thus from the condition (24) we obtain

$$
[g_k^i]_y = \sum_{j=1}^{6} \frac{\partial y_i}{\partial x_j}[g_k^j]_x,
$$

for $i, k = 1, \ldots, 6$. Hence for each vector field evaluated in $p = (0,0,0,0,0,0)$ we have in a matrix form

$$\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix} = [g_1]_y = \begin{pmatrix}
\frac{\partial y_1}{\partial x_1} - \frac{\sqrt{3}}{2}\frac{\partial y_1}{\partial x_4} + \frac{\sqrt{3}}{2}\frac{\partial y_1}{\partial x_6}\\
\frac{\partial y_2}{\partial x_1} - \frac{\sqrt{3}}{2}\frac{\partial y_2}{\partial x_4} + \frac{\sqrt{3}}{2}\frac{\partial y_2}{\partial x_6}\\
\frac{\partial y_3}{\partial x_1} - \frac{\sqrt{3}}{2}\frac{\partial y_3}{\partial x_4} + \frac{\sqrt{3}}{2}\frac{\partial y_3}{\partial x_6}\\
\frac{\partial y_4}{\partial x_1} - \frac{\sqrt{3}}{2}\frac{\partial y_4}{\partial x_4} + \frac{\sqrt{3}}{2}\frac{\partial y_4}{\partial x_6}\\
\frac{\partial y_5}{\partial x_1} - \frac{\sqrt{3}}{2}\frac{\partial y_5}{\partial x_4} + \frac{\sqrt{3}}{2}\frac{\partial y_5}{\partial x_6}\\
\frac{\partial y_6}{\partial x_1} - \frac{\sqrt{3}}{2}\frac{\partial y_6}{\partial x_4} + \frac{\sqrt{3}}{2}\frac{\partial y_6}{\partial x_6}
\end{pmatrix},$$

$$\begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix} = [g_2]_y = \begin{pmatrix}
\frac{\partial y_1}{\partial x_2} + \frac{1}{2}\frac{\partial y_1}{\partial x_4} - \frac{\partial y_1}{\partial x_5} + \frac{1}{2}\frac{\partial y_1}{\partial x_6}\\
\frac{\partial y_2}{\partial x_2} + \frac{1}{2}\frac{\partial y_2}{\partial x_4} - \frac{\partial y_2}{\partial x_5} + \frac{1}{2}\frac{\partial y_2}{\partial x_6}\\
\frac{\partial y_3}{\partial x_2} + \frac{1}{2}\frac{\partial y_3}{\partial x_4} - \frac{\partial y_3}{\partial x_5} + \frac{1}{2}\frac{\partial y_3}{\partial x_6}\\
\frac{\partial y_4}{\partial x_2} + \frac{1}{2}\frac{\partial y_4}{\partial x_4} - \frac{\partial y_4}{\partial x_5} + \frac{1}{2}\frac{\partial y_4}{\partial x_6}\\
\frac{\partial y_5}{\partial x_2} + \frac{1}{2}\frac{\partial y_5}{\partial x_4} - \frac{\partial y_5}{\partial x_5} + \frac{1}{2}\frac{\partial y_5}{\partial x_6}\\
\frac{\partial y_6}{\partial x_2} + \frac{1}{2}\frac{\partial y_6}{\partial x_4} - \frac{\partial y_6}{\partial x_5} + \frac{1}{2}\frac{\partial y_6}{\partial x_6}
\end{pmatrix},$$

$$\begin{pmatrix}0\\0\\1\\0\\0\\0\end{pmatrix} = [g_3]_y = \begin{pmatrix}
\frac{\partial y_1}{\partial x_3} - 2\frac{\partial y_1}{\partial x_4} - 2\frac{\partial y_1}{\partial x_5} - 2\frac{\partial y_1}{\partial x_6}\\
\frac{\partial y_2}{\partial x_3} - 2\frac{\partial y_2}{\partial x_4} - 2\frac{\partial y_2}{\partial x_5} - 2\frac{\partial 2_1}{\partial x_6}\\
\frac{\partial y_3}{\partial x_3} - 2\frac{\partial y_3}{\partial x_4} - 2\frac{\partial y_3}{\partial x_5} - 2\frac{\partial y_3}{\partial x_6}\\
\frac{\partial y_4}{\partial x_3} - 2\frac{\partial y_4}{\partial x_4} - 2\frac{\partial y_4}{\partial x_5} - 2\frac{\partial y_4}{\partial x_6}\\
\frac{\partial y_5}{\partial x_3} - 2\frac{\partial y_5}{\partial x_4} - 2\frac{\partial y_5}{\partial x_5} - 2\frac{\partial y_5}{\partial x_6}\\
\frac{\partial y_6}{\partial x_3} - 2\frac{\partial y_6}{\partial x_4} - 2\frac{\partial y_6}{\partial x_5} - 2\frac{\partial y_6}{\partial x_6}
\end{pmatrix},$$

$$\begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix} = [g_4]_y = \begin{pmatrix}
\frac{\partial y_1}{\partial x_4} + \frac{\partial y_1}{\partial x_5} + \frac{\partial y_1}{\partial x_6}\\
\frac{\partial y_2}{\partial x_4} + \frac{\partial y_2}{\partial x_5} + \frac{\partial y_2}{\partial x_6}\\
\frac{\partial y_3}{\partial x_4} + \frac{\partial y_3}{\partial x_5} + \frac{\partial y_3}{\partial x_6}\\
\frac{\partial y_4}{\partial x_4} + \frac{\partial y_4}{\partial x_5} + \frac{\partial y_4}{\partial x_6}\\
\frac{\partial y_5}{\partial x_4} + \frac{\partial y_5}{\partial x_5} + \frac{\partial y_5}{\partial x_6}\\
\frac{\partial y_6}{\partial x_4} + \frac{\partial y_6}{\partial x_5} + \frac{\partial y_6}{\partial x_6}
\end{pmatrix},$$

$$\begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix} = [g_5]_y = \begin{pmatrix}
\frac{\partial y_1}{\partial x_4} - 2\frac{\partial y_1}{\partial x_5} + \frac{\partial y_1}{\partial x_6}\\
\frac{\partial y_2}{\partial x_4} - 2\frac{\partial y_2}{\partial x_5} + \frac{\partial y_2}{\partial x_6}\\
\frac{\partial y_3}{\partial x_4} - 2\frac{\partial y_3}{\partial x_5} + \frac{\partial y_3}{\partial x_6}\\
\frac{\partial y_4}{\partial x_4} - 2\frac{\partial y_4}{\partial x_5} + \frac{\partial y_4}{\partial x_6}\\
\frac{\partial y_5}{\partial x_4} - 2\frac{\partial y_5}{\partial x_5} + \frac{\partial y_5}{\partial x_6}\\
\frac{\partial y_6}{\partial x_4} - 2\frac{\partial y_6}{\partial x_5} + \frac{\partial y_6}{\partial x_6}
\end{pmatrix},$$

$$\begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix} = [g_6]_y = \begin{pmatrix}-\sqrt{3}\frac{\partial y_1}{\partial x_4} + \sqrt{3}\frac{\partial y_1}{\partial x_6}\\ -\sqrt{3}\frac{\partial y_2}{\partial x_4} + \sqrt{3}\frac{\partial y_2}{\partial x_6}\\ -\sqrt{3}\frac{\partial y_3}{\partial x_4} + \sqrt{3}\frac{\partial y_3}{\partial x_6}\\ -\sqrt{3}\frac{\partial y_4}{\partial x_4} + \sqrt{3}\frac{\partial y_4}{\partial x_6}\\ -\sqrt{3}\frac{\partial y_5}{\partial x_4} + \sqrt{3}\frac{\partial y_5}{\partial x_6}\\ -\sqrt{3}\frac{\partial y_6}{\partial x_4} + \sqrt{3}\frac{\partial y_6}{\partial x_6}\end{pmatrix}.$$

We get a system of 36 PDEs. To solve them we group together 6, each containing a particular y_i. We demonstrate computation of y_1 which is composed of first rows of previous matrices

$$\begin{aligned}
1 &= \frac{\partial y_1}{\partial x_1} - \frac{\sqrt{3}}{2}\frac{\partial y_1}{\partial x_4} + \frac{\sqrt{3}}{2}\frac{\partial y_1}{\partial x_6},\\
0 &= \frac{\partial y_1}{\partial x_2} + \frac{1}{2}\frac{\partial y_1}{\partial x_4} - \frac{\partial y_1}{\partial x_5} + \frac{1}{2}\frac{\partial y_1}{\partial x_6},\\
0 &= \frac{\partial y_1}{\partial x_3} - 2\frac{\partial y_1}{\partial x_4} - 2\frac{\partial y_1}{\partial x_5} - 2\frac{\partial y_1}{\partial x_6},\\
0 &= \frac{\partial y_1}{\partial x_4} + \frac{\partial y_1}{\partial x_5} + \frac{\partial y_1}{\partial x_6},\\
0 &= \frac{\partial y_1}{\partial x_4} - 2\frac{\partial y_1}{\partial x_5} + \frac{\partial y_1}{\partial x_6},\\
0 &= -\sqrt{3}\frac{\partial y_1}{\partial x_4} + \sqrt{3}\frac{\partial y_1}{\partial x_6}.
\end{aligned}$$

This system of PDEs is in the matrix form

$$\begin{pmatrix}1&0&0&-\frac{\sqrt{3}}{2}&0&\frac{\sqrt{3}}{2}\\0&1&0&\frac{1}{2}&-1&\frac{1}{2}\\0&0&1&-2&-2&-2\\0&0&0&1&1&1\\0&0&0&1&-2&1\\0&0&0&-\sqrt{3}&0&\sqrt{3}\end{pmatrix}\begin{pmatrix}\frac{\partial y_1}{\partial x_1}\\\frac{\partial y_1}{\partial x_2}\\\frac{\partial y_1}{\partial x_3}\\\frac{\partial y_1}{\partial x_4}\\\frac{\partial y_1}{\partial x_5}\\\frac{\partial y_1}{\partial x_6}\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}. \tag{25}$$

Taking inverse matrix and multiplying (25) from left we get

$$\begin{pmatrix}\frac{\partial y_1}{\partial x_1}\\\frac{\partial y_1}{\partial x_2}\\\frac{\partial y_1}{\partial x_3}\\\frac{\partial y_1}{\partial x_4}\\\frac{\partial y_1}{\partial x_5}\\\frac{\partial y_1}{\partial x_6}\end{pmatrix} = \begin{pmatrix}1&0&0&0&0&-\frac{1}{2}\\0&1&0&0&-\frac{1}{2}&0\\0&0&1&2&0&0\\0&0&0&\frac{1}{3}&\frac{1}{6}&-\frac{\sqrt{3}}{6}\\0&0&0&\frac{1}{3}&-\frac{1}{3}&0\\0&0&0&\frac{1}{3}&\frac{1}{6}&\frac{\sqrt{3}}{6}\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}. \tag{26}$$

Final step to find y_1 is to integrate the following equation which we obtain from (26)

$$\frac{\partial y_1}{x_1} = 1.$$

Hence the result is

$$y_1 = x_1 + c(x_1).$$

We utilize the same procedure to find the rest of y_i. And finally we get a linear transformation of the following form

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & -\frac{1}{2} & 0 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{2} & 0 & 0 & -\frac{\sqrt{3}}{6} & 0 & \frac{\sqrt{3}}{6} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}. \tag{27}$$

With this adapted frame we can continue computing an approximation.

4.4 Nilpotent approximation

Let us consider vector fields g_1, g_2, g_3 in a new coordinate system (27). These vector fields are of order ≥ -1, hence it has in y coordinates a Taylor expansion

$$g_i(y) \sim \sum_{\alpha,j} a_{\alpha,j} y^\alpha \partial_{y_j},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex and $j = 1, \ldots, 6$. Moreover, we define a weighted degree of the monomial $y^\alpha = y_1^{\alpha_1} \ldots y_n^{\alpha_n}$ as $w(\alpha) = w_1\alpha_1 + \cdots + w_n\alpha_n$, therefore

$$w(\alpha) \geq w_j - 1 \qquad \text{if} \quad a_{\alpha,j} \neq 0.$$

If we group together the monomial vector fields of same weighted degree we can write g_i as a series

$$g_i = g_i^{(-1)} + g_i^{(0)} + g_i^{(1)} + \ldots,$$

where $g_i^{(s)}$ is a homogeneous vector field of degree s. Let us recall that the following equality holds

$$w_j = \mathrm{ord}_p(y_j).$$

In our case the coordinate weights are $(1, 1, 1, 2, 2, 2)$.

We also recall that a Lie algebra $Lie(X_1, \dots, X_m)$ is said to be *nilpotent of step* s if all brackets X_I of lenght $|I|$ greater than s are zero.

Hence we set the approximation $\hat{g}_i = g_i^{(-1)}$, $i = 1, 2, 3$. The family of vector fields $(\hat{g}_1, \hat{g}_2, \hat{g}_3)$ is a first-order approximation of (g_1, g_2, g_3) at p and generates a nilpotent Lie algebra of step $r = w_n = 1$. It follows that every bracket of the vector fields $\hat{g}_1, \hat{g}_2, \hat{g}_3$ is zero if the length is greater then 1. Thus in our particular case we have

$$\begin{aligned}
\hat{g}_1 &= \partial_{y_1} - \frac{1}{2} y_2 \partial_{y_4} + \left(-\frac{1}{2} y_2 - y_3 \right) \partial_{y_5} - \frac{1}{2} y_1 \partial_{y_6}, \\
\hat{g}_2 &= \partial_{y_2} + \frac{1}{2} y_1 \partial_{y_4} - \frac{1}{2} y_1 \partial_{y_5} + \left(\frac{1}{2} y_2 - y_3 \right) \partial_{y_6}, \\
\hat{g}_3 &= \partial_{y_3}.
\end{aligned}$$

The family $(\hat{g}_1, \hat{g}_2, \hat{g}_m)$ is called the *nilpotent approximation* of (g_1, g_2, g_3) at p associated with the coordinates y.

We obtain the remaining three vector fields by taking Lie brackets of $(\hat{g}_1, \hat{g}_2, \hat{g}_m)$. Hence we get

$$\begin{aligned}
\hat{g}_4 &= [\hat{g}_1, \hat{g}_2] = \partial_{y_4}, \\
\hat{g}_5 &= [\hat{g}_1, \hat{g}_3] = \partial_{y_5}, \\
\hat{g}_5 &= [\hat{g}_2, \hat{g}_3] = \partial_{y_6}.
\end{aligned}$$

We verify the fact that our approximation is nilpotent of step 1 in Maple. In other words, we compute $[\hat{g}_i, [\hat{g}_j, \hat{g}_k]]$ for $i, j, k = 1, 2, 3$ and verify that every combination gives 0. In fact in the Maple code is shown the result for the most complicated vector fields. The rest is omitted because one can show that Lie bracket of $[\partial_z, \partial_v]$ is zero for arbitrary z, v.

> $with(DifferentialGeometry) : with(LieAlgebras) :$

> $DGsetup([y_1, y_2, y_3, y_4, y_5, y_6], M, verbose) :$

The following coordinates have been protected:

$$[y_1, y_2, y_3, y_4, y_5, y_6]$$

The following vector fields have been defined and protected:

$$[D\_y_1, D\_y_2, D\_y_3, D\_y_4, D\_y_5, D\_y_6]$$

The following differential 1-forms have been defined and protected:

$$[dy_1, dy_2, dy_3, dy_4, dy_5, dy_6] \quad \textbf{(1)}$$

M > $g_{1,\,h} := D\_y_1 - \frac{y_2}{2} D\_y_4 + \left(-\frac{y_2}{2} - y_3\right) D\_y_5 - \frac{y_1}{2} D\_y_6$

$$g_{1,\,h} := D\_y_1 - \frac{1}{2}\, y_2\, D\_y_4 + \left(-\frac{1}{2}\, y_2 - y_3\right) D\_y_5 - \frac{1}{2}\, y_1\, D\_y_6 \quad \textbf{(2)}$$

M > $g_{2,\,h} := D\_y_2 + \frac{y_1}{2} D\_y_4 - \frac{y_1}{2} D\_y_5 + \left(\frac{y_2}{2} - y_3\right) D\_y_6$

$$g_{2,\,h} := D\_y_2 + \frac{1}{2}\, y_1\, D\_y_4 - \frac{1}{2}\, y_1\, D\_y_5 + \left(\frac{1}{2}\, y_2 - y_3\right) D\_y_6 \quad \textbf{(3)}$$

M > $g_{3,\,h} := D\_y_3$

$$g_{3,\,h} := D\_y_3 \quad \textbf{(4)}$$

M > $g_{4,\,h} := LieBracket(g_{1,\,h}, g_{2,\,h})$

$$g_{4,\,h} := D\_y_4 \quad \textbf{(5)}$$

M > $g_{5,\,h} := LieBracket(g_{1,\,h}, g_{3,\,h})$

$$g_{5,\,h} := D\_y_5 \quad \textbf{(6)}$$

M > $g_{6,\,h} := LieBracket(g_{2,\,h}, g_{3,\,h})$

$$g_{6,\,h} := D\_y_6 \quad \textbf{(7)}$$

M > $LieBracket(g_{1,\,h}, g_{4,\,h});$
$LieBracket(g_{1,\,h}, g_{5,\,h});$
$LieBracket(g_{1,\,h}, g_{6,\,h});$
$LieBracket(g_{2,\,h}, g_{4,\,h});$
$LieBracket(g_{2,\,h}, g_{5,\,h});$
$LieBracket(g_{2,\,h}, g_{6,\,h});$

$$0\, D\_y_1$$
$$0\, D\_y_1$$
$$0\, D\_y_1$$
$$0\, D\_y_1$$
$$0\, D\_y_1$$
$$0\, D\_y_1 \quad \textbf{(8)}$$

5 Motion planning

This chapter introduce a basic overview of motions of the robot described by vector fields on the manifold M. In this section we introduce motions based on the original controlling vector fields. For more details look at [6],[9],[12].

5.1 Vector field motions

In the previous text we introduced three control vector fields for our model

$$g_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \\ \sin(\phi_1 - \frac{2}{3}\pi) \\ \sin(\phi_2) \\ \sin(\phi_3 + \frac{2}{3}\pi) \end{pmatrix}, \quad g_2 = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \\ -\cos(\phi_1 - \frac{2}{3}\pi) \\ -\cos(\phi_2) \\ -\cos(\phi_3 + \frac{2}{3}\pi) \end{pmatrix}, \quad g_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1-\cos(\phi_1) \\ -1-\cos(\phi_2) \\ -1-\cos(\phi_3) \end{pmatrix}.$$

We also derived their Lie brackets in this form

$$g_{12} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad g_{23} = \begin{pmatrix} \sin\theta \\ \cos\theta \\ 0 \\ \frac{1}{2} - \cos(\phi_1 - \frac{2}{3}\pi) \\ -1-\cos(\phi_2) \\ \frac{1}{2} - \cos(\phi_1 + \frac{2}{3}\pi) \end{pmatrix}, \quad g_{13} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \\ -\frac{\sqrt{3}}{2} + \sin(\phi_1 - \frac{2}{3}\pi) \\ \sin(\phi_2) \\ \frac{\sqrt{3}}{2} + \sin(\phi_3 + \frac{2}{3}\pi) \end{pmatrix}.$$

Our Lie algebra generated by vector fields g_1, g_2, g_3 and their Lie brackets g_{12}, g_{23}, g_{13} is in the following form

$$\bar{G} = \text{span}\{g_1, g_2 g_3, g_{12}, g_{23}, g_{13}\}. \tag{28}$$

To describe motions for each vector field we need to set a initial point $q_0 = (x, y, \theta, \phi_1, \phi_2, \phi_3) = (0, 0, 0, 0, 0, 0)$. Evaluating the controllability Lie algebra (28) at the origin of the state space, we have

$$\bar{G}(\mathbf{0}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & -2 & 1 & 1 & -\sqrt{3} \\ 0 & -1 & -2 & 1 & -2 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & -2 & 1 & 1 & \sqrt{3}. \end{pmatrix} \tag{29}$$

Let us decribe a motion appropriate to each vector field. This part is based on [6] and [12].

The effect of g_1 motion is the following. The body of snake moves along the x–axis and legs change proportionally to $-1 : 0 : 1$. It means that the first leg moves in the negative–clockwise direction and the third vice versa.

$$g_1(\mathbf{0}) = \left(1, 0, 0, -\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}\right)$$

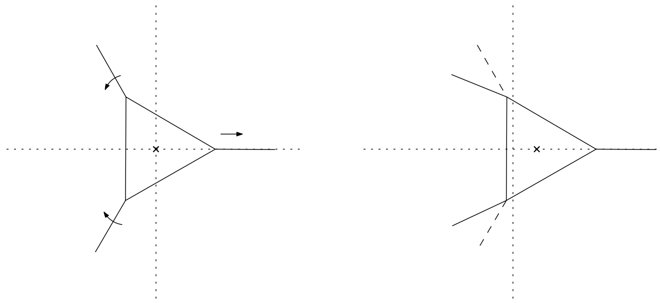

Figure 8: Effect of g_1 motion

The second vector field indicates a motion along the y–axis and rotaion of all legs proportionally to $1 : -2 : 1$. It means that the first and the third legs rotate in a anti–clockwise direction and the second in the clockwise direction.

$$g_2(\mathbf{0}) = \left(0, 1, 0, \frac{1}{2}, -1, \frac{1}{2}\right)$$

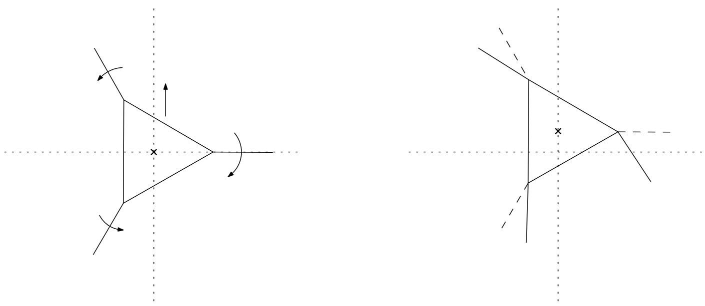

Figure 9: Effect of g_2 motion

As we can see in Figure 10 the effect of g_3 motion is pure rotation. All legs rotate in the same direction (clockwise) proportionally to $1 : 1 : 1$. And the body of the robot also rotates but in the opposite direction.

$$g_3(\mathbf{0}) = (0, 0, 1, -2, -2, -2)$$

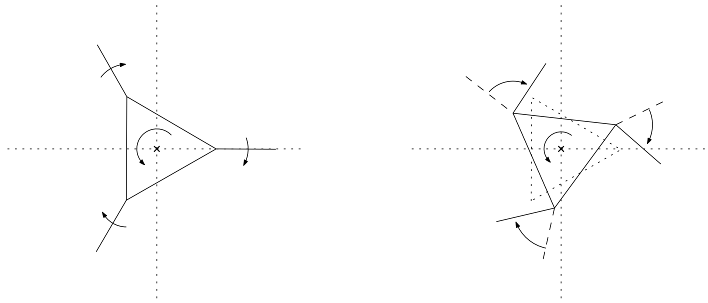

Figure 10: Effect of g_3 motion

Next step is to describe the realization of Lie bracket motions. Here we state the final result of the motion but the way how to realize it is much more complicated and will be discussed in the next section by using periodic input. Hence an important remark is that each Lie bracket motion is in fact a combination of two basic vector fields.

Thus the final result of a g_{12} motion is the following. The configuration of a robot is not changed, only legs are rotated in the clockwise direction proportionally to $1:1:1$.

$$g_{12}(\mathbf{0}) = (0,0,0,1,1,1)$$

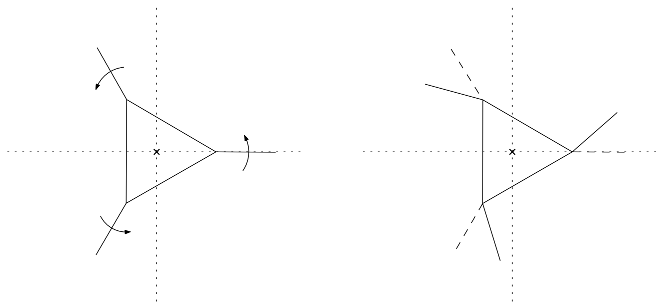

Figure 11: Effect of g_{12} motion

The effect of g_{13} motion can be described as follows. Trident snake robot moves along y–axis and all legs rotate. The first and third legs rotate in the positive direction and the second vice versa. Moreover the propotion is $1:-2:1$.

$$g_{13}(\mathbf{0}) = (0,1,0,1,-2,1)$$

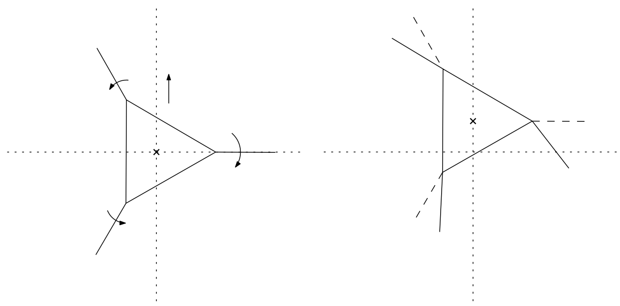

Figure 12: Effect of g_{13} motion

The realization of the g_{23} motion has the following effect. The body of a robot moves along x–axis and the first and third legs move against each other.

$$g_{23}(\mathbf{0}) = \left(1, 0, 0, -\sqrt{3}, 0, \sqrt{3}\right)$$

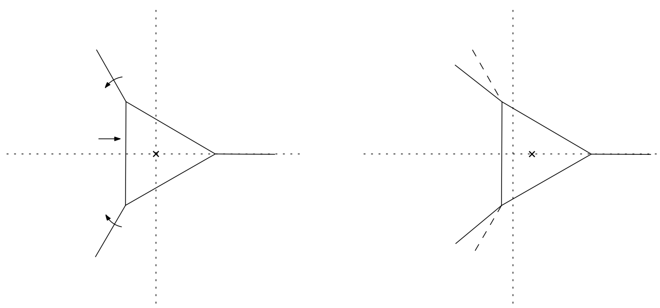

Figure 13: Effect of g_{23} motion

5.2 Translation and rotation

This section introduce three basic motions on a planar surface. They are

- motion along x–axis
- motion along y–axis
- rotation around z–axis.

Note that a purpose of rotation control is to change the orientation of the robot θ without changing the position x, y and the shape ϕ. The same idea applies to a translation in both directions. We change the position without changing the orientation and the shape.

Using a combination of these motions we are able to move the robot to a required position on a planar surface.

Motion of robot in x direction means to go from $q_{init} = (0,0,0,0,0,0)^T$ to $q_{final} = (1,0,0,0,0,0)^T$. We will achieve that using a combination of vector field g_1 and Lie bracket vector field g_{23} in the following form

$$2g_{23}(0) - g_1(0) = (1,0,0,0,0,0)^T$$

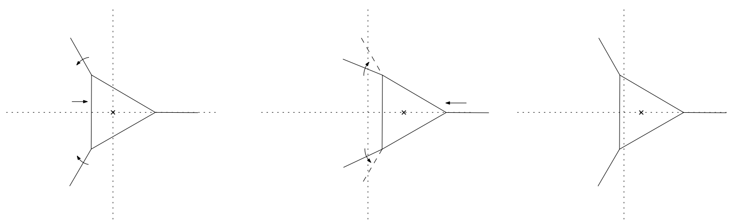

Figure 14: Translation in x–direction

Motion of robot in y direction means to go from $q_{init} = (0,0,0,0,0,0)^T$ to $q_{final} = (0,1,0,0,0,0)^T$. We will achieve that using a combination of vector field g_2 and Lie bracket vector field g_{12} in the following form

$$2g_2(0) - g_{13}(0) = (0,1,0,0,0,0)^T$$

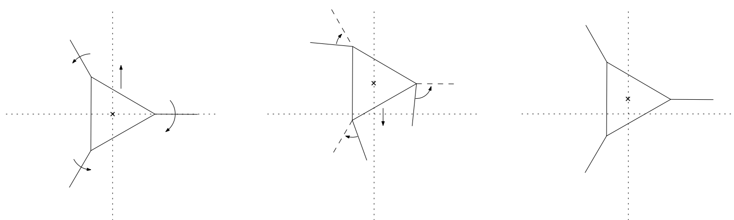

Figure 15: Translation in y–direction

Rotation of a trident snake robot means in a sense of vector field motions to travel from $q_{init} = (0,0,0,0,0,0)^T$ to $q_{final} = (0,0,1,0,0,0)^T$. Rotation is realized by combination of g_{12} and g_3 motion

$$2g_{12}(0) + g_3(0) = (0,0,1,0,0,0)^T.$$

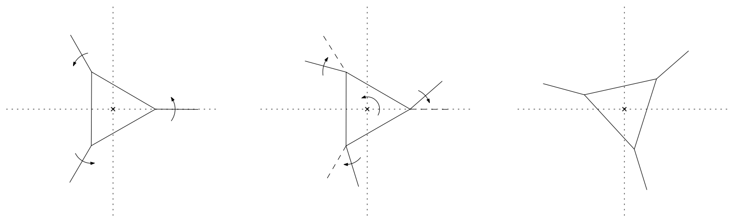

Figure 16: Rotation

Now we have an overview of basic motions which can be realized to control a trident snake robot. Next step will be to simulate them in a suitable software and to compare original vector fields and their nilpotent approximation.

6 Simulation in V-REP

For a simulation of a basic motions of our robot we use software named V-REP (Virtual Robot Experimentation Platform). For more information about this software look at [2] or [1]. The main property of this environment is that it includes physical properties and influences of surrounding environment.

Model of a trident snake robot is composed of several basic objects (body, legs, wheels and joints) which can be controled by scripts. In our case we control legs using actuated joints (servomotors) placed between each leg and a body. Therefore the whole mechanism is controled by these three joints.

6.1 Periodic input

As we mentioned in the previous text Lie bracket motion is a specific combination of two incoming vector fields. According to [6] and [9] we represent this motion as a result of periodic input with sufficiently small amplitude.

Lie bracket motions can be realized using periodic input in the following form

$$
\begin{aligned}
v(t) &= (-A\omega \sin(\omega t), A\omega \cos(\omega t), 0)^T && \text{for } g_{12}, \\
v(t) &= (-A\omega \sin(\omega t), 0, A\omega \cos(\omega t))^T && \text{for } g_{13}, \\
v(t) &= (0, -A\omega \sin(\omega t), A\omega \cos(\omega t))^T && \text{for } g_{23},
\end{aligned}
$$

where $A \in \mathbb{R}$ is a positive amplitude and $\omega \in \mathbb{N}$ is a frequency.

Thus the first Lie bracket motion is realized with values $A = 0.3$, $\omega = 3$[rad/s] hence the evolution of a deflection of each leg ϕ_i is the following

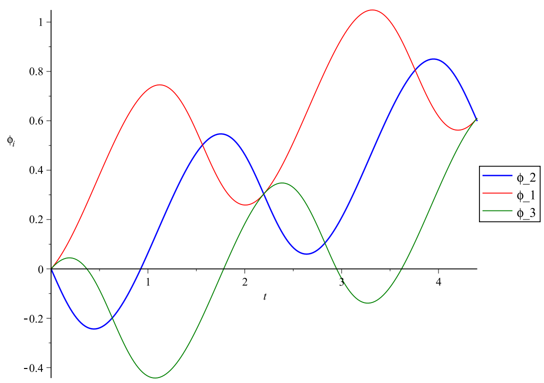

Figure 17: Evolution of ϕ_i for g_{12} motion

Let us show the effect of the second Lie bracket vector field g_{13}. We set $A = 0.2$ and $\omega = 3$[rad/s] thus we obtain the result depicted in Figure 18.

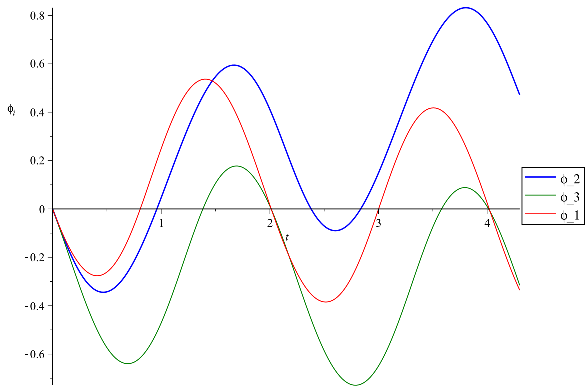

Figure 18: Evolution of ϕ_i for g_{13} motion

Finally, the evolution of leg angles appropriate to g_{23} motion is presented with values $A = 0.25$ and $\omega = 2$[rad/s].

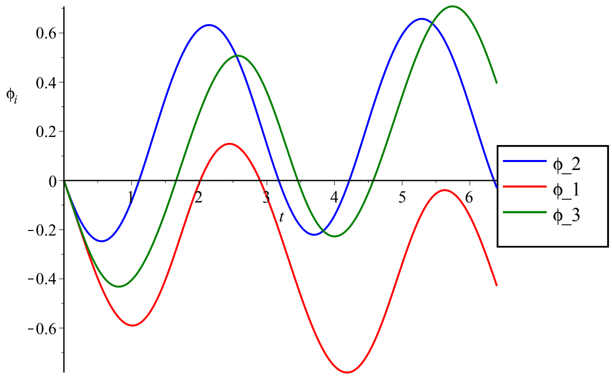

Figure 19: Evolution of ϕ_i for g_{23} motion

Note that each graph visualizes two cycles (periods) of the motion.

6.2 Simulation

In this section we introduce our results of simulating the basic motions in V–REP. Some of these simulations are shown in the video record attached to this text in Appendix. We omit the simulation of g_1, g_2 and g_3 and visualize more complicated ones.

During the computations and modeling motions of a trident snake robot inaccuracies may appear. They are caused especially by applying periodic input as a realization of Lie brackets, using inexact numerical calculations to evaluate solutions of systems of ODEs in Maple, a physical environment and free version of software V–REP which can contains simplier or inexact numerical solver. All these circumstances can cause differences between simulated and theoretical motions.

Let us introduce the first simulated motion g_{12} during time $t \in (0, 2T)$ where T is a period of a periodic input. Here we present three graphs which represent the process. The first graph shows time response of a position of a body center, second graph shows the rotation θ of the body and the third shows trajectory of the body center on the x–y plane.

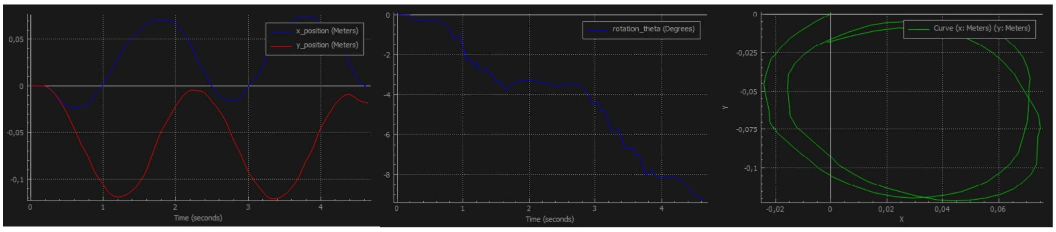

Figure 20: Realization of g_{12} motion

Notice that g_{12} causes rotation in a negative (clockwise) direction and the body follows small circular path anti–clockwise. In Figure 21 the process is shown from the initial position to the final one.

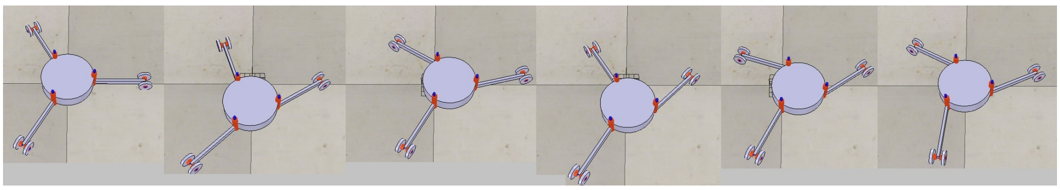

Figure 21: Realization of g_{12} motion

Let us continue with presentation of the results achieved from our simulations. We introduce g_{13} motion using graphs depicted in Figure 22. As we already know from the previous text applying the periodic input appropriate to g_{13} a trident snake robot moves along y–axis and the rotation of a body θ in the final position approaches zero. Note that the simulation is realized in time interval $t \in (0, 2T)$ where T is a period of a periodic input.

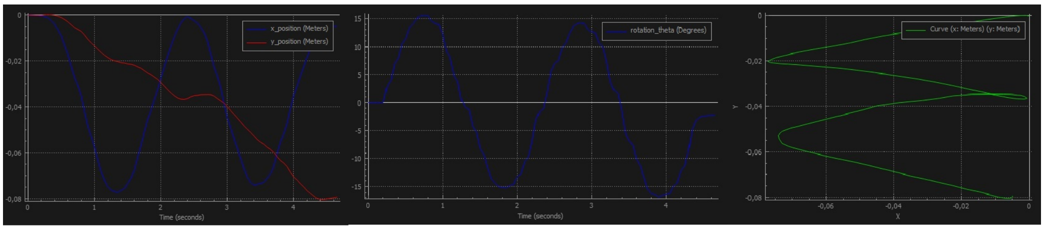

Figure 22: Realization of g_{13} motion

To achieve a better view we introduce a sequence of positions of a trident snake robot to illustrate the simulation process of g_{13} motion in Figure 23.

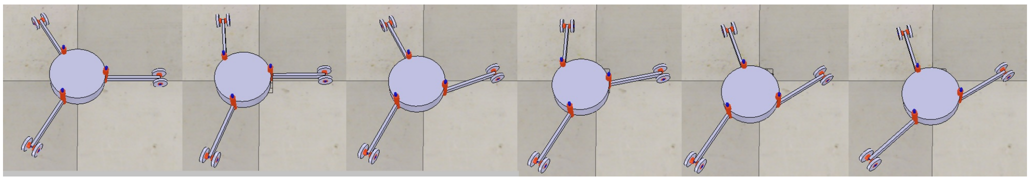

Figure 23: Realization of g_{13} motion

Since g_{23} motion causes a translation in the x-axis direction we observe this trend in our graphs in Figure 24. After two cycles (i.e. $t \in (0, 2T)$) rotation angle of the body approaches zero as well as the y position of the center. However, the body moves along x–axis.

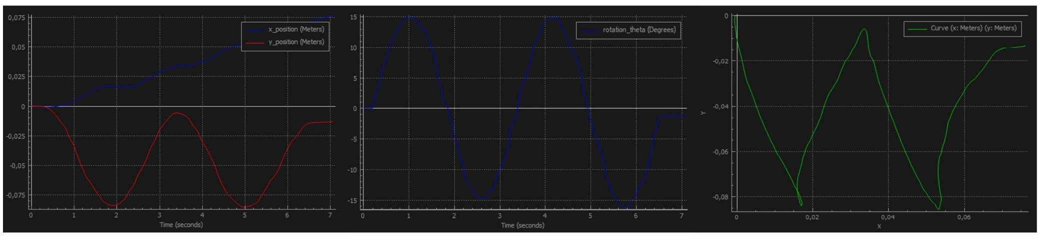

Figure 24: Realization of g_{23} motion

We also present a sequence of positions in Figure 25 which leads to the final position of a trident snake after applying g_{23} motion.

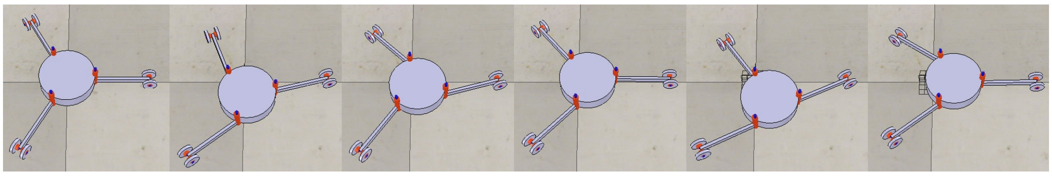

Figure 25: Realization of g_{23} motion

6.3 Comparison of models

We computed a nilpotent approximation of a controlling distribution in the previous text. However after applying this approximated model to V–REP we realized that the result is almost identical to the original one which was expected according to the results in [5]. Moreover because of all the inaccuracies during computations for our simulations the difference between approximated model and the original one disappeared. Hence due to the previous reasons we note that V–REP is not suitable for comparing approximated and the original model. Vizualization of the difference computed in MATLAB may be seen in [5].

Conclusion

We described a nonholonomic system called trident snake robot. We derived kinematic equations corresponding to the fact that the motion is limited by constrains on passive wheels. We introduced this model in a sense of differential geometry notions and discussed the controllability of this system. Due to the fact that the dimension of a state space of the system is the same as the dimension of a controlling distribution of this system we came to the fact that trident snake robot is locally controllable.

Then we tried to simplify our model. Usually a linearization of a dynamical model is reached by the theory of equillibrium points and Jacobi matrices. But it is not suitable for nonholonomic systems. Therefore we introduced a sub-Riemannian distance of two points associated with a nonholonomic system which can be understood as a minimal time needed for the nonholonomic system to go from the first point to the second with bounded controls. Hence in a sense of this new distance we provided an algorithm leading to the creation of an approximation of the system. New notions like nonholonomic order of a function/vector field at a point, privileged coordinates were neccesary to define. For better understanding we also showed some examples.

We obtained a nilpotent approximation which means that the Lie algebra generated by approximated vector fields $\hat{g}_1, \hat{g}_2, \hat{g}_3$ is nilpotent because all Lie brackets of length greater than 1 are zero which we showed in Maple.

Next part of this text presented an overview of motions caused by controlling vector fields. We also presented how to obtain a pure translation (in x an y direction) and rotation around z-axis. By combining these basic motions we can reach arbitrary point of our state space.

Finally we introduced results of simulations in an environment called V–REP. We simulated more complicated motions appropriate to the Lie brackets of vector fields which were realized by periodic input. But note that it brought some inaccuracy to this simulation.

Finally, we mentioned comparison of our models. We found out that V-REP is not suitable for comparing the original and approximated model. Because the simulation is influenced by many inaccuracies, therefore we are not able to distiguish which inaccuracy came from the aprroxiamtion and which is from the simulation. The choice of a suitable tool is still an open question, yet we mention several sources dealing with this topic.

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Electronic Appendix Index

1. trident_simulation.ttt - V-REP script
2. g_12_motion.avi - video of realization g_{12} motion
3. g_13_motion.avi - video of realization g_{13} motion
4. g_23_motion.avi - video of realization g_{23} motion